Degenerations of special holonomy metrics

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Special holonomy

10.92 Corollary. *Let (M, g) be a Riemannian manifold of dimension n which is not locally symmetric and whose holonomy representation* Hol^0 *is irreducible. Then its holonomy representation* Hol^0 *is one of the following*:

(I) $\mathrm{Hol}^0 = SO(n)$

(II) $n = 2m$ and $\mathrm{Hol}^0 = U(m) \quad (m \geqslant 2)$

(III) $n = 2m$ and $\mathrm{Hol}^0 = SU(m) \quad (m \geqslant 2)$

(IV) $n = 4m$ and $\mathrm{Hol}^0 = Sp(1) \cdot Sp(m) \quad (m \geqslant 2)$

(V) $n = 4m$ and $\mathrm{Hol}^0 = Sp(m) \quad (m \geqslant 2)$

~~(VI) $n = 16$ and $\mathrm{Hol}^0 = \mathrm{Spin}(9)$~~

(VII) $n = 8$ and $\mathrm{Hol}^0 = \mathrm{Spin}(7)$

(VIII) $n = 7$ and $\mathrm{Hol}^0 = G_2$.

Hyperkähler 4-manifolds

- A Riemannian 4-manifold (M^4, g) is **hyperkähler** if $\mathrm{Hol}(g) \subseteq \mathrm{SU}(2)$.
- **hyperkähler triple**: $\underline{\boldsymbol{\omega}} = (\omega_1, \omega_2, \omega_3)$ 2-forms on M such that

$$d\omega_i = 0, \qquad \tfrac{1}{2}\,\omega_i \wedge \omega_j = \delta_{ij}\,\omega_1^2.$$

- Hyperkähler 4-manifolds are **Kähler Ricci-flat**:
  - $\omega_c = \omega_2 + i\omega_3$, $\omega_c \wedge \omega_c = 0$, $\omega_c \wedge \overline{\omega}_c > 0 \rightsquigarrow$ complex structure J
  - Kähler form $\omega = \omega_1$ ($\omega \wedge \omega_c = 0$, $d\omega = 0$) with vanishing Ricci-form

$$\omega^2 = \tfrac{1}{2}\omega_c \wedge \overline{\omega}_c$$

- The (smooth 4-manifold underlying a complex) **K3 surface**
  - every Einstein metrics on the K3 surface is hyperkähler

$$\frac{1}{2\pi^2}\int_M \tfrac{1}{48}\mathrm{Scal}^2 + |W_+|^2 = 2\chi + 3\tau = 0, \qquad \triangle_{\Lambda^+} = \nabla^*\nabla - 2W_+ + \tfrac{1}{3}\mathrm{Scal}$$

  - **Period map**

$$\mathcal{P}\colon\ \mathcal{M} \longrightarrow \mathrm{Gr}^+(3,19)/\mathrm{Aut}(\mathrm{H}^2(K3,\mathbb{Z})), \qquad g_{\underline{\boldsymbol{\omega}}} \longmapsto \mathrm{span}([\underline{\boldsymbol{\omega}}]).$$

Non-collapsed limits

Theorem (Nakajima 1988, Bando–Kasue–Nakajima 1989, Anderson 1990)

Fix $\Lambda, C, V, D > 0$ and let (M_i^4, g_i) be a sequence of Einstein 4-manifolds satisfying

1. $|\mathrm{Ric}(g_i)| \leq \Lambda$,
2. $\chi(M_i) \leq C$,
3. $\mathrm{Vol}(M_i, g_i) \geq V$,
4. $\mathrm{diam}(M_i, g_i) \leq D$.

Then a subsequence converges to an Einstein **orbifold** (M_∞, g_∞) with a **definite number** of isolated singular points.

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Then a subsequence converges to an Einstein **orbifold** (M_∞, g_∞) with a **definite number** of isolated singular points.

- **non-collapsing**: $\mathrm{Vol}_{g_i}(B_1(p)) \geq v > 0$ for all $p \in M_i$
- **finite energy**: $\|\mathrm{Rm}_{g_i}\|_{L^2} \leq C$

$$8\pi^2\chi(M) = \int_M \tfrac{1}{24}\mathrm{Scal}^2 + |W|^2 - \tfrac{1}{2}|\mathring{\mathrm{Ric}}|^2$$

Non-collapsed limits

Theorem (continued)

$(M_i^4, g_i) \xrightarrow{GH} (M_\infty, g_\infty)$ Einstein orbifold
For each singular point $x \in M_\infty$ one can find $x_i \in M_i$ and $r_i \to \infty$ such that, up to subsequences, $x_i \to x$ and $(M_i, r_i^2 g_i, x_i)$ converges to a Ricci-flat **ALE** 4-manifold (W, h, x_∞) of rate $\nu = -4$.

- $\Gamma \subset SO(4)$ finite, acting freely on $\mathbb{R}^4 \setminus \{0\}$
- $f\colon (\mathbb{R}^4 \setminus B_R)/\Gamma \to W \setminus K$

$$|\nabla^k(f^*h - h_{\mathbb{R}^4/\Gamma})| = O(r^{\nu-k})$$

- Bando (1990), Anderson–Cheeger (1991): "bubble-tree" of ALE orbifolds

ALE gravitational instantons

- **gravitational instanton**: complete hyperkähler 4-manifold with decaying curvature
  - finite energy: $\|\mathrm{Rm}\|_{L^2} < \infty$
  - faster than quadratic curvature decay: $|\mathrm{Rm}| = O(r^{-2-\delta})$, $\delta > 0$
- **volume growth**: $c\, r \leq \mathrm{Vol}(B_r) \leq C\, r^4$
- Bando–Kasue–Nakajima (1989): complete Ricci-flat $d = 4$ + finite energy + maximal volume growth $\Longrightarrow$ ALE of rate -4

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- **Eguchi–Hanson metric** on $T^*S^2 = \mathcal{O}_{\mathbb{P}^1}(-2)$ (1978)

$$\varphi_t = \sqrt{r^4 + t^4} + 2t^2 \log r - t^2 \log\left(\sqrt{r^4 + t^4} + t^2\right)$$

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- Kronheimer (1989)
  - Γ finite group of SU(2) that acts freely on $\mathbb{C}^2 \setminus \{0\}$
  - X_Γ smooth 4-manifold underlying minimal resolution of $\mathbb{C}^2/\Gamma$
  - $[\underline{\omega}](\Sigma) \neq \underline{\mathbf{0}} \in \mathbb{R}^3$ for all $\Sigma \in H_2(X_\Gamma, \mathbb{Z})$, $\Sigma \cdot \Sigma = -2$

The Kummer construction

Gibbons–Pope 1979, LeBrun–Singer 1994, Donaldson 2012

Building blocks:

- A background metric to model the region where curvature stays bounded
  Flat orbifold $T^4/\mathbb{Z}_2$; cut-out neighbourhoods of the 16 orbifold singularities
- Non-compact "bubbles" to model the geometry of high curvature regions
  Eguchi–Hanson ALE metric on T^*S^2

Gluing:

- Use the building blocks to construct an approximate solution.
- Use analysis to perturb to an exact Ricci-flat metric.

The Kummer construction

- Complex Monge–Ampère equation
  - construct complex structure by blow-up: ω_c
  - solve $(\omega + i\partial\overline{\partial}u)^2 = \frac{1}{2}\omega_c \wedge \overline{\omega}_c$ using the Implicit Function Theorem
- Gluing hyperkähler triples
  - choice of gauge $\mathrm{SO}(4)/\mathrm{U}(2) \simeq S^2$ at each singular point
  - gluing $\rightsquigarrow$ closed approximately hyperkähler triple $\underline{\boldsymbol{\omega}}_t$

$$\omega_i \wedge \omega_j = Q_{ij}\,\omega_1^2, \qquad Q = \mathrm{id} + O(t^2)$$

  - perturb $\underline{\boldsymbol{\omega}}_t \mapsto \underline{\boldsymbol{\omega}}_t + d\underline{\mathbf{a}} + \underline{\boldsymbol{\zeta}}$, $\underline{\mathbf{a}} \in \Omega^1$, $\underline{\boldsymbol{\zeta}} \in \mathcal{H}^+ \otimes \mathbb{R}^3$,

$$d^+\underline{\mathbf{a}} + \underline{\boldsymbol{\zeta}} = \mathcal{F}(Q - \mathrm{id} + d^-\underline{\mathbf{a}} * d^-\underline{\mathbf{a}}), \qquad d^*\underline{\mathbf{a}} = 0$$

  - parameter count: $58 = 10 + 16 \times 3$

Higher dimensions

- **codimension 4 orbifold singularities**
  - Codimension 4 Conjecture: Cheeger–Naber (2015), Cheeger (2003), Cheeger–Tian (2005)
  - Joyce's construction of G_2 and $Spin_7$ manifolds by desingularising T^d/Γ (1996)
- **isolated conical singularities**
  - Riemannian cones with special holonomy (Sasaki–Einstein, 3–Sasakian, nearly Kähler manifolds,...)
  - deformation and smoothing theory (Joyce, Chan, Karigiannis, Karigiannis–Lotay), AC Calabi–Yau manifolds (Joyce, vanCoevering, Goto, Conlon–Hein)
  - Hein–Sun (2017): existence of Calabi–Yau manifolds with isolated conical singularities

Collapse

- (M^4, g_i), $\mathrm{Ric}(g_i) = 0$, $\mathrm{Vol}(g_i) = 1$, $\mathrm{diam}(g_i) \to \infty$

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- Anderson (1992)

Theorem III below.) Thus, suppose that for all $x \in M$,

$$\mathrm{vol}_{g_i}(B_x(1)) \to 0 \,, \quad \text{as } i \to \infty \,. \tag{5.2}$$

In this case, we have the following result. Recall that a sequence of Riemannian manifolds (X_i, g_i) collapses, in the sense of Cheeger-Gromov [15], if

$$\begin{aligned} &\mathrm{inj}_i(x) \to 0 \,, \quad \text{as } i \to \infty \,, \text{ and} \\ &\mathrm{inj}_i(x)^2 \cdot |R_i|(x) \le \varepsilon_0 \,, \end{aligned} \tag{5.3}$$

for any $x \in X_i$. Here $\mathrm{inj}_i(x)$ is the injectivity radius of (X_i, g_i) at x, while ε_0 is a small absolute constant (the Cheeger-Gromov constant).

THEOREM 5.1. *If $\{g_i\}$ is a sequence of Einstein metrics on M, of volume 1, satisfying (5.2), then $\{g_i\}$ collapses, in the sense of Cheeger-Gromov, metrically on the complement of finitely many points $\{z_k\} \in M$.*

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- Cheeger–Tian (2006): **bounded curvature outside finitely many points**

Theorem 0.1 (Collapse implies L_2 concentration of curvature). *There exists $v > 0$, β, c, such that the following holds. Let M^4 denote a complete Einstein 4-manifold satisfying*

$$|\lambda| \leq 3\,, \tag{0.2}$$

$$\int_{M^4} |R|^2 \leq C\,, \tag{0.3}$$

and for all p and some $s \leq 1$,

$$\frac{\mathrm{Vol}(B_s(p))}{s^4} \leq v\,. \tag{0.4}$$

Then there exist $p_1, \ldots, p_N$, with

$$N \leq \beta \cdot C\,, \tag{0.5}$$

such that

$$\int_{M^4 \setminus (\bigcup_i B_s(p_i))} |R|^2 \leq c \cdot \left(\sum_i \frac{\mathrm{Vol}(B_s(p_i))}{s^4} + \lim_{r \to \infty} \frac{\mathrm{Vol}(B_r(p))}{r^4} \right)\,. \tag{0.6}$$

Theorem 0.8 (ϵ-regularity). *There exists $\epsilon > 0$, c, such that the following holds. Let M^4 denote an Einstein 4-manifold satisfying* (0.2) *and let $r \leq 1$. If $B_s(p)$ has compact closure for all $s \leq r$ and*

$$\int_{B_r(p)} |R|^2 \leq \epsilon\,, \tag{0.9}$$

then

$$\sup_{B_{\frac{1}{2}r}(p)} |R| \leq c \cdot r^{-2}\,. \tag{0.10}$$

Collapse

- Anderson (1992): Cheeger–Gromov collapse outside finitely many points
- Cheeger–Tian (2006): bounded curvature outside finetely many points
- Cheeger–Fukaya–Gromov (1992): collapse with bounded curvature
  - **locally**: $3r \in (0, \|\mathrm{Rm}_{g_i}\|_{L^\infty}^{-\frac{1}{2}})$
  - $\left(B_{3r}(0) \subset T_{p_i} M_i \simeq \mathbb{R}^n, \exp_{p_i}^* g_i\right) \xrightarrow{C^\infty} (B_{3r}(0), \hat{g}_\infty)$
  - local pseudo-group of isometries Γ_i of $(B_r(0), \exp_{p_i}^* g_i)$: $x \sim_{\Gamma_i} y \Longleftrightarrow \exp_{p_i}(x) = \exp_{p_i}(y) \in M_i$
  - $\Gamma_i \to \Gamma_\infty$ pseudo-group of isometries of $(B_r(0), \hat{g}_\infty)$ with neighbourhood of 1 isomorphic to neighbourhood of 1 in nilpotent Lie group
  - $(B_r(p_i) \subset M_i, g_i) \xrightarrow{\mathrm{GH}} (B_r(0), \hat{g}_\infty)/\Gamma_\infty$

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  - $(B_r(p_i) \subset M_i, g_i) \xrightarrow{\mathrm{GH}} (B_r(0), \hat{g}_\infty)/\Gamma_\infty$
- Page (1981): Kummer construction along a 1–parameter family of "split" 4-tori $T^4 = T^{4-k} \times T^k_\epsilon$ with $\mathrm{Vol}(T^k_\epsilon) = \epsilon \to 0 \rightsquigarrow$ complete Ricci-flat manifolds asymptotic to $(\mathbb{R}^{4-k} \times T^k)/\mathbb{Z}_2$ as rescaled limits
  Hitchin (1984), Biquard–Minerbe (2011)

The Gibbons–Hawking Ansatz

The **Gibbons–Hawking Ansatz** (1978): local form of hyperkähler metrics in dimension 4 with a triholomorphic circle action

- h **positive harmonic function** on $U \subset \mathbb{R}^3$
- $M \to U$ principal U(1)–bundle and connection θ with $d\theta = *dh$

$$g = h\, g_{\mathbb{R}^3} + h^{-1}\theta^2 \text{ is a hyperkähler metric on } M$$

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Example: ALE and ALF metrics of cyclic type

$$g_m = \left(m + \sum_{i=1}^{n} \frac{1}{2|x - a_i|} \right) dx \cdot dx + \left(m + \sum_{i=1}^{n} \frac{1}{2|x - a_i|} \right)^{-1} \theta^2$$

- $a_1, \ldots, a_n$ distinct $\rightsquigarrow$ complete metric
  $a_1 = \cdots = a_{k+1} \rightsquigarrow$ orbifold singularity $\mathbb{C}^2/\mathbb{Z}_k$
- m is called the mass
  - $m = 0 \rightsquigarrow$ **ALE**
  - $m > 0 \rightsquigarrow$ **ALF**

ALF gravitational instantons

- (M^4, g) is **ALF** (asymptotically locally flat):
  finite group $\Gamma < O(3)$ acting freely on S^2
  $M \setminus K \to (\mathbb{R}^3 \setminus B_R)/\Gamma$ circle fibration and (up to a double cover)

$$|\nabla^k (g - g_\infty)| = O(r^{-\tau-k}), \qquad \tau > 0, \qquad g_\infty = g_{\mathbb{R}^3/\Gamma} + \theta^2$$

  - $\Gamma = \mathrm{id} \implies$ ALF space of **cyclic** type
  - $\Gamma = \mathbb{Z}_2 \implies$ ALF space of **dihedral** type

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Minerbe (2010–2011)

- (M^4, g), $\mathrm{Ric}(g) \geq 0$, quadratic curvature decay, $\mathrm{Vol}\big(B_r(p)\big) \leq Cr^a$ with $a < 4$ for all $p \in M \implies a \leq 3$
- cubic volume growth + faster than quadratic curvature decay + hyperkähler $\implies$ ALF
- all ALF spaces of cyclic type obtained from Gibbons–Hawking ansatz
  - multi-Taub–NUT space with $n+1$ "nuts"
  - A_n ALF space

ALF spaces of dihedral type

(M, g) is a D_m ALF space if up to a double cover g is asymptotic to the Gibbons–Hawking metric obtained from the harmonic function

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G. Chen – X. Chen (2015) classified ALF spaces of dihedral type:

- The D_0 ALF space is the Atiyah–Hitchin manifold (1988).
- The D_1 ALF metrics are the double cover of the Atiyah–Hitchin metric and its Dancer deformations (1993).
- The D_2 ALF spaces are Page's "periodic but nonstationary" gravitational instantons. Constructed by Hitchin (1984) and Biquard–Minerbe (2011).
- D_m, $m \geq 3$, constructed by Cherkis–Kapustin (1999) and Cherkis–Hitchin (2005), Biquard–Minerbe (2011) and Auvray (2012).

Codimension 1 collapse

Theorem (F. 2016)

For every collection of 8 ALF spaces of dihedral type $M_1, \dots, M_8$ and n ALF spaces of cyclic type $N_1, \dots, N_n$ satisfying

$$\sum_{j=1}^{8} \chi(M_j) + \sum_{i=1}^{n} \chi(N_i) = 24$$

there exists a sequence $\{g_\epsilon\}$ of Ricci-flat metrics on the K3 surface s.t.:

- As $\epsilon \to 0$ the metric g_ϵ collapses to the flat orbifold $\mathbb{T}^3/\mathbb{Z}_2$ with bounded curvature outside $8 + n$ points
- An ALF space of dihedral type arises as a rescaled limit of the sequence close to one of the fixed points of the involution on $\mathbb{T}^3$
- An ALF space of cyclic type arises as a rescaled limit of the sequence close to one of the other n points

The GH ansatz over a punctured 3-torus

- flat 3-torus + involution τ with $\mathrm{Fix}(\tau) = \{q_1, \ldots, q_8\}$
  - integer weight $m_j \in \mathbb{Z}_{\geq 0}$ to each q_j
- further distinct $2n$ points $\pm p_1, \ldots, \pm p_n$
  - integer weight $k_i \geq 1$ to each pair $\pm p_i$
- $\sum m_j + \sum k_i = 16 \Longrightarrow$ harmonic function h with prescribed singularities

$$h = \frac{k_i}{2\,\mathrm{dist}(\pm p_i, \cdot)} + O(1) \qquad h = \frac{2m_j - 4}{2\,\mathrm{dist}(q_j, \cdot)} + O(1)$$

- (incomplete) hyperkähler metric

$$g_\epsilon^{\mathrm{gh}} = (1 + \epsilon\, h)\, g_{T^3} + \epsilon^2 (1 + \epsilon\, h)^{-1} \theta^2$$

- for $\epsilon > 0$ sufficiently small $1 + \epsilon\, h > 0$ outside of balls of radius $\propto \epsilon$ around the points q_j with $m_j = 0, 1$
- glue in
  - an $A_{k_i - 1}$ ALF space close to $\pm p_i$
  - a D_{m_j} ALF space close to q_j
- perturb resulting approximate hyperkähler triple

Higher dimensions

Collapse of 7-dimensional G_2–manifolds to Calabi–Yau 3-folds

Theorem (F.–Haskins–Nordström 2017)

Let $(B, g_0, \omega_0, \Omega_0)$ be an **asymptotically conical Calabi–Yau 3-fold** asymptotic to a Calabi–Yau cone (C, g_C) and let $M \to B$ be a **principal circle bundle**.

Assume that $c_1(M) \neq 0$ but $c_1(M) \cup [\omega_0] = 0$.

Then for every $\epsilon > 0$ sufficiently small there exists an **$\mathbf{S}^1$–invariant $\mathbf{G}_2$–holonomy metric** g_ϵ on M with:

- ALF–type asymptotics: as $r \to \infty$, $g_\epsilon = g_C + \epsilon^2 \theta_\infty^2 + O(r^{-\nu})$
- collapses with bounded curvature as $\epsilon \to 0$: $g_\epsilon \sim_{C^{k,\alpha}} g_0 + \epsilon^2 \theta^2$

Collapse along elliptic fibrations

- $\pi\colon (M, \omega_c) \to \mathbb{P}^1$ **elliptic** complex K3 surface (with a section)
- 24 fibres of Kodaira type I_1 (pinched tori)
- $[\omega_\epsilon] \cdot \pi^{-1}(z) = \varepsilon$

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- **semi-flat metric** $\omega_{sf,\varepsilon}$ away from singular fibres
  - $\omega^2_{sf,\varepsilon} = \frac{1}{2}\omega_c \wedge \overline{\omega}_c$
  - $\omega_{sf,\varepsilon}|_{\pi^{-1}(z)}$ flat metric of volume ε
- **Ooguri–Vafa metric** in the neighbourhood of singular fibres
  - GH ansatz on $B_r(0)\times S^1 \subset \mathbb{C}_z \times S^1$

$$h = -\tfrac{1}{2\pi\varepsilon}\log|z| + \sum_{m\in\mathbb{Z}^*}\tfrac{1}{2\pi\varepsilon}e^{it}K_0\left(\tfrac{2\pi}{\varepsilon}|mz|\right)$$

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- Approximate solution ω_ε

$$\omega_\varepsilon^2 = \left(1 + O(e^{-c/\varepsilon})\right)\omega_c \wedge \overline{\omega}_c$$

- complex Monge–Ampère $\left(\omega_\varepsilon^2 + i\partial\overline{\partial}u_\varepsilon\right)^2 = \omega_c \wedge \overline{\omega}_c$
  - all constants in Yau's proof blow-up polynomially in ε^{-1}

Collapse along elliptic fibrations

Theorem (Gross–Wilson 2000)

Let $\pi\colon (M, \omega_c) \to \mathbb{P}^1$ be an elliptic complex K3 surface (with a section) with 24 I_1 singular fibres. As $\varepsilon \to 0$ the Kähler Ricci-flat metric ω_ε such that $[\omega_\varepsilon] \cdot \pi^{-1}(z) = \varepsilon$ satisfies:

1. For every $k \geq 2$, $\alpha \in (0, 1)$ and every simply connected set $U \subset \mathbb{P}^1$ with closure contained in the complement of the 24 points corresponding to singular fibres there exist constants $C, c > 0$ such that $\|u_\varepsilon\|_{C^{k,\alpha}(U)} \leq Ce^{-c/\varepsilon}$.
2. (M, ω_ε) converges in Gromov–Hausdorff sense to $\mathbb{P}^1$ endowed with the distance induced by a metric ω_0 defined away from the 24 singular points and satisfying $\mathrm{Ric}(\omega_0) = \omega_{WP}$.

Gross–Tosatti–Zhang (2013, 2016): extension of this result to arbitrary elliptic complex K3 surfaces

ALG and ALH gravitational instantons

Hein 2012

- $\pi\colon X \to \mathbb{P}^1$ rational elliptic surface
- ω_c on $M = X \setminus \pi^{-1}(\infty)$
- Kähler metric ω on M with $\omega = \omega_{\mathsf{sf}}$ at infinity
- complex Monge–Ampère equation on $M \rightsquigarrow$ complete hyperkähler metric on M with volume growth r^2 (ALG), $r^{\frac{4}{3}}$ or r (ALH)

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Examples with faster than quadratic curvature decay

- Biquard–Minerbe (2011): minimal resolution of $(E \times \mathbb{C})/\Gamma$ (ALG) or $(\mathbb{R} \times T^3)/\mathbb{Z}_2$ (ALH with linear volume growth)
- Chen–Chen (2015): classification of gravitational instantons with faster than quadratic curvature decay
- Chen–Chen (2015): ALH spaces with linear volume growth and "stretching-the-neck" degenerations

ALG and ALH gravitational instantons

Examples with quadratic curvature decay

- Gibbons–Hawking ansatz on $(\mathbb{R}^2_z \times S^1)/\mathbb{Z}_2$ or $(\mathbb{R}_s \times T^2)/\mathbb{Z}_2$

$$h = \tfrac{2b}{2\pi} \log |z|, \qquad h = \tfrac{2b}{2\pi} |s|$$

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Degenerations of complex K3 surfaces

- Kulikov model $\pi\colon \mathcal{X} \to \triangle$ ($\mathcal{X}$ smooth, $K_{\mathcal{X}} \simeq \mathcal{O}_{\mathcal{X}}$, $\pi^{-1}(0)$ reduced snc)
  - Type I: $\pi^{-1}(0)$ smooth
  - Type II: $\pi^{-1}(0)$ chain of $k \geq 2$ surfaces, rational surfaces at either end, elliptic ruled surfaces in the middle, double curves smooth elliptic curves
  - Type III: $\pi^{-1}(0)$ rational surfaces meeting along cycles of rational curves; dual graph is a triangulation of S^2
- after hyperkähler rotation Gross–Wilson is Type III
- Kobayashi (1990): speculations about metric realisation of degenerations using ALG and ALH spaces