The isoperimetric problem in general relativity

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1. Constant mean curvature surfaces into the euclidean space

2. CMC in the Riemannian setting, the role of scalar curvature

3. General Relativity framework: Asymptotic flatness, mass, center of mass

4. Huisken-Yau and Ye canonical foliations

5. Perspectives
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5. Perspectives
Let $\Sigma \subset \mathbb{R}^3$ be an oriented surface.
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**Definition**

The quadratic form on $T_p\Sigma$ defined by

$$II_p(\vec{v}) := -\langle d\tilde{N}_p(\vec{v}), \vec{v} \rangle,$$

is called the second fundamental quadratic form of $\Sigma$. 
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**Definition**

We define two maps $K$ et $H$, namely the Gauss and the mean curvature, as follows

$$K(p) = \det(d\vec{N}_p)$$

and

$$H(p) = \frac{1}{2} \text{trace}(d\vec{N}_p).$$
The isoperimetric problem in general relativity
Constant mean curvature surfaces into the euclidean space

Theorem
Surfaces which minimize their area with a fixed volume (isoperimetric surfaces) have constant mean curvature.
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Constant mean curvature surfaces into the euclidean space

\[ \vec{N}(p) \vec{v}(p) \]

\[ H = \kappa_1 + \kappa_2 \]
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Constant mean curvature surfaces into the euclidean space

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\[ H = \kappa_1 + \kappa_2^2 \text{ and } K = \kappa_1 \kappa_2 \]

A' \left( 0 \right) = -2 \int_{\Sigma} f H \, d\Sigma e \quad V' \left( 0 \right) = \int_{\Sigma} f \, d\Sigma \]

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*Surfaces which minimize their area with a fixed volume (isoperimetric surfaces) have constant mean curvature.*
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Constant mean curvature surfaces into the euclidean space
Classification of CMC into $\mathbb{R}^3$

Theorem (Hopf 1951)

*Let $S$ be a compact simply connected surface with constant mean curvature, then it is a round sphere.*
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**Theorem (Wente 1983)**

There exists an immersion of $T^2$ into $\mathbb{R}^3$ whose image has constant mean curvature.

Kapouleas proved in the 90’ that there are CMC surfaces of arbitrary genus.
Second variation of the area:

\[ A''(0) = - \int_{\Sigma} f(\Delta f + \|II\|^2 f) d\Sigma. \]
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A CMC is (weakly) **stable** if for all \( f \in C_c^\infty(\Sigma) \) (with \( \int f = 0 \)) then

\[ \int_{\Sigma} \|II\|^2 f^2 d\Sigma \leq \int_{\Sigma} \|\nabla f\|^2 d\Sigma \]
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**Theorem (Fischer-Colobrie, Schoen, 82, Barbosa, Do Carmo, 84)**

**The only (weakly) stable CMC (complete) surfaces of** \( \mathbb{R}^3 \) **are planes and round spheres.**
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CMC in the Riemannian setting, the role of scalar curvature

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CMC in the Riemannian setting, the role of scalar curvature

The pertubative setting

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**Theorem (Ye, 1991)**

Let \((\mathcal{N}, g)\) be a Riemannian manifold and \(p \in \mathcal{N}\) a non-degenerate critical point of the scalar curvature. Then there exists a surface with constant mean curvature in every neighborhood of \(p\).
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We can relax the hypothesis of non-degeneracy for existence: Pacard et Xu 2009.
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We can relax the hypothesis of non-degeneracy for existence: Pacard et Xu 2009.
This condition is also necessary, Laurain 2011.
Theorem (Johnson & Morgan, 2000)

Let $(\mathcal{N}, g)$ be a compact Riemannian manifold. Then isoperimetric domains of small volume are nearly round spheres.
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Let \((\mathcal{N}', g)\) be a compact Riemannian manifold. Then isoperimetric domains of small volume are nearly round spheres.

Theorem (Druet, 2002)

Let \((\mathcal{N}', g)\) be a compact Riemannian manifold and \(\Omega_V\) a sequence of isoperimetric domains of volume \(V\), then

\[\Omega_V \to p \text{ as } V \to 0,\]

where \(p\) is a point of maximum of the scalar curvature.
The second variation of area is:

\[ A''(0) = -\int_{\Sigma} f (\Delta f + (\|II\|^2 + Ricc(\vec{N}, \vec{N}))f) d\Sigma. \]
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Using Gauss equation it becomes

\[ A''(0) = - \int_{\Sigma} f \left( \Delta f + \left( \frac{1}{2} (\|II\|^2 + R - \frac{K}{2}) f \right) \right) \, d\Sigma, \]

where \( R \) is the scalar curvature and \( K \) the Gaussian curvature.
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$$A''(0) = -\int_{\Sigma} f \left( \Delta f + \left( \frac{1}{2}(\| II \|^2 + R - \frac{K}{2}) \right) f \right) d\Sigma,$$

where $R$ is the scalar curvature and $K$ the Gaussian curvature. Hence when the scalar curvature is non-negative we can derive some topological constraint on $\Sigma$. For instance, in the compact case, if $H = 0$ (stable) the genus is smaller than 1 (Schoen-Yau), if $H$ is large enough (weakly stable) the genus is smaller than 3 (Rosenberg).
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General Relativity postulates:

- The space-time is a (3, 1) Lorentzian Manifold ($\tilde{\mathcal{M}}, \tilde{g}$).
- Free particles travel along time-geodesic.
- $\tilde{g}$ satisfies the Einstein equation:

$$\tilde{\text{Ric}} - \frac{\tilde{R}}{2} \tilde{g} = 8\pi T,$$

where $T$ is energy momentum tensor.
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What are the possible Universe (space slice \((M, g)\))?

To simplify we consider some time-symmetric space-time which is isolated.
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What are the possible Universe (space slice \((M, g)\))? To simplify we consider some time-symmetric space-time which is isolated.

**So Mathematically the question is:**
What are the asymptotically flat 3-Riemannian manifolds with non-negative scalar curvature?
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General Relativity framework: Asymptotic flatness, mass, center of mass

**Definition**

Let \((M, g)\) be a 3-manifold, it is said to be Asymptotically Flat (AF) (with one end), if there exists a compact \(K\) such that \(M \setminus K\) is diffeomorphic to \(\mathbb{R}^3 \setminus B(0, 1)\) and in those coordinates

\[
g = \delta^{ij} + O_2(|x|^{-\tau}),
\]

with \(\tau > \frac{1}{2}\).
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g = \delta^{ij} + O_2(|x|^{-\tau}),
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with \(\tau > \frac{1}{2}\).

**Theorem (Arnowitt, Deser, Misner, 61, Bartnik, Chrusciel, 80')**

Let \((M, g)\) be an asymptotically flat manifold such that \(R \in L^1\), then the following limit exists

\[
\lim_{R \to +\infty} \frac{1}{16\pi} \int_{S(0,R)} (g_{ij,i} - g_{ii,j})\nu^j \ d\sigma,
\]

moreover it depends only on the metric. Let denote it \(m\) for mass.
Let $(M, g)$ be an AF manifold with nonnegative scalar curvature. Then $m \geq 0$ with equality if and only if $M$ is isometric to $\mathbb{R}^3$. 

**Theorem (Schoen-Yau, 79)**
The unique rotationally invariant solution of Einstein-Equation is given by the space slice $\mathbb{R}^3 \setminus \{0\}$, $\left(1 + \frac{m}{2|x|}\right)^4 \delta_{ij}$ is an AF manifold with vanishing scalar curvature and mass $m$. It is the Schwarzschild metric.
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$$g = \left(1 + \frac{m}{2|x|}\right)^4 \delta^{ij} + O_2(|x|^{-2}).$$

The mass is unchanged by translation, considering $\mathbb{R}^3 \setminus \{p\}, \left(1 + \frac{m}{2|x-p|}\right)^4 \delta_{ij}$.

But can we detect the "center" of this translated version of Schwarzschild?
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General Relativity framework: Asymptotic flatness, mass, center of mass

Definition

Let \((M, g)\) be an AF manifold, such that

\[
|g_{ij} - \delta_{ij}| + |x| |\Gamma^k_{ij}| + |x|^2 |Ric_{ij}| + |x|^\frac{5}{2} |S| \leq \frac{C}{|x|^\frac{1}{2} + \varepsilon}.
\]

Then it satisfies the weak Regge-Teitelboim condition, if

\[
|g(x) - g(-x)| + |x| |\Gamma(x) + \Gamma(-x)| \leq \frac{C}{|x|^{1+\varepsilon}}.
\]

It satisfies the strong Regge-Teitelboim condition, if

\[
|g(x) - g(-x)| + |x| |\Gamma(x) + \Gamma(-x)| + |x|^2 |Ric(x) - Ric(-x)| + |x|^\frac{5}{2} |S(x) - S(-x)| \leq \frac{C}{|x|^\frac{3}{2} + \varepsilon}.
\]
The isoperimetric problem in general relativity

General Relativity framework: Asymptotic flatness, mass, center of mass

Theorem (Beig, O’Murchadha, 90)

Let $(M, g)$ an AF manifold satisfying the strong RT condition, with non vanishing mass, then the following limit exists

$$
\lim_{R \to +\infty} \frac{1}{16\pi m} \int_{S(0, R)} (g_{ij,i} - g_{ii,j}) \nu^j x^\alpha - (g_{i\alpha} \nu^i - g_{ii\nu^\alpha}) \, d\sigma,
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moreover it depends only on the metric. Let denote it $C_{ADM}^\alpha$ for the center of mass.
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The strong RT condition has been proved to be optimal by Cederbaum & Nerz (13): Constructing metric with divergent center of mass.
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Huisken-Yau and Ye canonical foliations

Existence

Theorem (Christodoulo-Yau, 88)

Let \((M, g)\) be a 3-manifold with none-negative scalar curvature, then the Hawking quasi-local mass

\[
m(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}} \left(1 - \frac{1}{4\pi} \int_{\Sigma} H^2 d\sigma \right)
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of a closed stable CMC is non negative.
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Theorem (Huisken-Yau 96, Ye 97)

Let \(M\) be a Schwarzschildian manifold with positive mass, then for \(R\) large enough we can perturb the sphere \(S(0, R)\) into a stable CMC surface \(\Sigma_R\). Those spheres form a foliation.
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**Theorem (Huisken-Yau 96, Ye 97)**

Let \(M\) be a Schwarzschildian manifold with positive mass, then for \(R\) large enough we can perturb the sphere \(S(0, R)\) into a stable CMC surface \(\Sigma_R\). Those spheres form a foliation.

Improvements: L.H. Huang, J. Metzger and finally C. Nerz(14) who prove the existence into an AF manifold.
Using this foliation $\Sigma_R$, you can take the following limit

$$C_{HY}^{\alpha} = \lim_{R \to \infty} \frac{\int_{\Sigma_R} x^{\alpha} \, d\sigma}{\int_{\Sigma_R} d\sigma},$$

As a new definition of center of mass.
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$$C_{HY} = C_{ADM}.$$

The foliation provide also some kind of intrinsic coordinates, what about uniqueness?
Theorem (Qing, Tian, 07)

Let $(M, g)$ a Schwarzschildian manifold with positive mass. Then exists a compact set $K$ such that stable CMC spheres which separates the infinity from $K$ coincide with the leafs of the CMC foliation.

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Uniqueness

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Improvements:
Ma (10) AF+ Strong RT, Ma(16) $|g - \delta| = O_4(r^{-1})$ and $|S| = O(r^{-3+\varepsilon})$, Laurain-Metzger(17 under the weak RT.

A sequence of spheres which does not separate a compact set from infinity either intersect a compact region or is outlying.
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Global uniqueness

Theorem (Carlotto, Chodosh and Eichmair, 16)

Let \((M, g)\) be a complete 3 Manifold, Schwarzschildian with positive mass and with none negative curvature. Then for every compact \(K\) there exists \(\alpha_K > 0\) such that if \(\Sigma\) is a stable constant mean curvature surfaces then, with \(|\Sigma| \geq \alpha_K\), then \(\Sigma\) is disjoint from \(K\).
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Consequence: to get global uniqueness, we need to exclude the outlying case.
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It is a consequence of the following quantitative version of the Positive Mass Theorem

**Theorem (Carlotto, Chodosh and Eichmair, 16)**

Let \((M, g)\) be a complete 3 Manifold, Schwarzschildian with positive mass and with none negative curvature and which has horizon boundary. The the only complete stable minimal embeddings are embeddings of components of the horizon.
The assumption on the positive mass and the Schwarzschildian asymptotic flatness can’t be drop due to the following theorem
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**Theorem (Carlotto, Schoen, 16)**

*Given a scalar-flat asymptotically flat metric $g$ there exist cones and scalar-flat asymptotically flat metrics which coincides with $g$ inside of the cones and are flat outside slightly larger cones.*
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Here the decay of metric is $\tau \in (1/2, 1)$. 
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Theorem (Brendle, Eichmair, 14)

There is a complete Riemannian 3-manifold \((M, g)\) that is Schurschildean with mass \(m > 0\) which admits a sequence of arbitrary large outlying stable constant mean curvature surfaces \(\Sigma_k\).
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Outlying sphere

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**Theorem (Brendle and Eichmair, 14)**

Let $(M, g)$ be a complete Riemannian 3-manifold that is Schwarschildean with mass $m > 0$ in the following sense

\[ g_{ij} = \left(1 + \frac{m}{2r}\right)^4 \delta_{ij} + T_{ij} + o_4(r^{-2}) \]

where $T$ is an homogeneous tensor of degree $-2$. If the scalar curvature satisfy $R \geq -o(r^{-4})$, there is no sequence of outlying stable constant mean curvature surfaces $\Sigma_k$ such that

\[ \lim_{k \to \infty} d(\Sigma_k, 0) H_k \in (0, \infty) . \]
Theorem (Chodosh and Eichmair, 17)

Let \((M, g)\) be a complete 3-Manifold, Schwarzschildian with positive mass and with none negative curvature. We fix \(K\) a compact set, then there exists \(\eta > 0\), such that for every outlying stable constant mean curvature surface \(\Sigma\), we have

\[d(\Sigma, K)H_\Sigma \geq \eta.\]
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\[ d(\Sigma, K) H_\Sigma \geq \eta. \]

To get global uniqueness it suffices to exclude outlying surface whose distance to origin is much bigger than the diameter.
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Outlying sphere

Theorem (Chodosh and Eichmair, 17)

Let \((M, g)\) be a complete 3-Manifold, Schwarschildean with positive mass and with none negative curvature, which satisfies either

\[ R \equiv 0 \]

or

\[ x^i x^j \partial_i \partial_j R \geq 0 \text{ outside a compact set} \]

Then any stable constant mean curvature surface with area large enough is part of the canonical foliation.
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Theorem (O. Chodosh, M. Eichmair, Y. Shi, and H. Yu, 16)

Let $(M, g)$ be a complete Riemannian 3-manifold that is asymptotically flat and which has non-negative scalar curvature. Unless $(M, g)$ is isometric to flat $\mathbb{R}^3$, for every sufficiently large $V > 0$, there is a unique surface of least area that encloses volume $V$ in $(M, g)$. This surface is a leaf of the canonical foliation.
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The tools to prove this theorem is the study of the behavior of the Hawking mass, which is in a certain sense encode the defect of the isoperimetric ratio, this idea was introduce by Bray.
Theorem (Bray, 98)

*In the exact Schwarzschild geometry, with $m > 0$, the spherically symmetric spheres minimize the area among all other surfaces in their homology class containing the same volume (separating them from the horizon).*
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*In the exact Schwarzschild geometry, with \( m > 0 \), the spherically symmetric spheres minimize the area among all other surfaces in their homology class containing the same volume (separating them from the horizon).*

**Theorem (Brendle, 13)**

*Every closed embedded constant mean curvature surface in the exact Schwarzschild, with \( m > 0 \), geometry is a spherically symmetric sphere.*
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Theorem (Bray, 98)

In the exact Schwarzschild geometry, with $m > 0$, the spherically symmetric spheres minimize the area among all other surfaces in their homology class containing the same volume (separating them from the horizon).

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Every closed embedded constant mean curvature surface in the exact Schwarzschild, with $m > 0$, geometry is a spherically symmetric sphere.

Note that this result does not require the surfaces to be large or stable. It can be seen as a very general version of the Alexandrov theorem, since it also holds to be true in some general wrapped product with non-negative "curvature".
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- A new quasi-local mass?
Thank you for your attention!