Some smooth applications of non-smooth Ricci curvature lower bounds
1^{st} Part: non-smooth Ricci curvature lower bounds

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Some notational remarks on the curvature bounds

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- Examples:
  - n-dimensional euclidean space: $\mathrm{Sec} \equiv 0$, $\mathrm{Ric} \equiv 0$.
  - n-dimensional round sphere of radius 1: $\mathrm{Sec} \equiv 1$, $\mathrm{Ric} \equiv n-1$.
  - n-dimensional hyperbolic space: $\mathrm{Sec} \equiv -1$, $\mathrm{Ric} \equiv -(n-1)$.

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- Upper/Lower bounds on the sectional curvature are strong assumptions with strong implications E.g. Cartan-Hadamard Theorem (if $\mathrm{Sec} \leq 0$ then the universal cover of M is diffeomorphic to $\mathbb{R}^N$), Topogonov triangle comparison theorem($\rightsquigarrow$ definition of Alexandrov spaces: non smooth spaces with upper/lower bounds on the "sectional curvature"), etc.

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- Upper bounds on the Ricci curvature are very (too) weak assumption for geometric conclusions. E.g. Lokhamp Theorem (Gao-Yau, Brooks in dim 3): any closed manifold of dim$\geq$ 3 carries a metric with negative Ricci curvature.

Some basics of comparison geometry: lower Ricci bounds

Lower bounds on the Ricci curvature: natural framework for comparison geometry

- Bishop-Gromov volume comparison: (not most general form)
  If (M^n, g) has $\mathrm{Ric} \geq 0$ then for all $x \in M$

$$R \mapsto \frac{\mathrm{vol_g}(B_R(x))}{\omega_n R^n} \text{ is monotone non-increasing}$$

- Laplacian comparison,
- Cheeger-Gromoll splitting,
- Li-Yau inequalities on heat flow,
- Levy-Gromov isoperimetric inequality,
- ...

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• Natural question: what can we say about the compactification of the space of Riemannian manifolds with Ricci curvature bounded below (by, say, minus one)?
•Hope: useful also to establish properties for smooth manifolds.

Semi-smooth setting

- Cheeger-Colding 1997-2000 three fundamental papers on JDG on the structure of Ricci limit spaces.
  - Collapsing: $\lim_k vol_{g_k}(B_1(\bar{x}_k)) = 0 \rightsquigarrow$ loss of dimension in the limit. More difficult, nevertheless they proved that the limit space has a uniquely defined volume measure (up to scaling) and a.e. point has a euclidean tangent space (the dimension may vary from point to point). Such points are called regular points, the complementary is called singular set.

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  $\rightsquigarrow$ weak version of a Riemannian manifold with Ric$\geq K$; analogy with GMT (currents, varifolds,etc.)

Preliminary Observation

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- Ricci curvature is a property of lenghts and volumes: needs also a reference volume measure
  $\rightsquigarrow$ natural setting metric measure spaces $(X, \mathrm{d}, \mathfrak{m})$.

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- Given $\mu_1, \mu_2 \in \mathcal{P}(X)$, define the (Kantorovich-Wasserstein) quadratic transportation distance

$$W_2^2(\mu_1, \mu_2) := \inf \left\{ \int_{X \times X} \mathrm{d}^2(x, y)\, \gamma(dxdy) \right\}$$

where $\gamma \in \mathcal{P}(X \times X)$ with $(\pi_i)_\sharp \gamma = \mu_i$, $i = 1, 2$

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where $\gamma \in \mathcal{P}(X \times X)$ with $(\pi_i)_\sharp \gamma = \mu_i$, $i = 1, 2$
- $(\mathcal{P}(X), W_2)$ is a metric space, geodesic if (X, d) is geodesic .

Non smooth setting 2: Entropy functionals

- On the metric space $(\mathcal{P}(X), W_2)$ consider the Entropy functionals $\mathcal{U}_{N,\mathfrak{m}}(\mu)$ if $\mu << \mathfrak{m}$

$$\mathcal{U}_{N,\mathfrak{m}}(\rho\mathfrak{m}) := -N \int \rho^{1-\frac{1}{N}} d\mathfrak{m} \quad \text{if } 1 < N < \infty \quad \text{Reny Entropy}$$

$$\mathcal{U}_{\infty,\mathfrak{m}}(\rho\mathfrak{m}) := \int \rho \log \rho d\mathfrak{m} \quad \text{Shannon Entropy}$$

(if μ is not a.c. then if $N < \infty$ the non a.c. part does not contribute, if $N = +\infty$ then set $\mathcal{U}_{\infty,\mathfrak{m}}(\mu) = \infty$.)

Non smooth setting: intrinsic-axiomatic definition. 2

- Crucial observation
  [CorderoErausquin-McCann-Schmuckenshlager '01, Otto-Villani '00, Sturm-Von Renesse '05]
  If $(X, \mathsf{d}, \mathfrak{m})$ is a smooth Riemannian manifold (M, g), then $\mathrm{Ric} \geq 0$ (resp. $\geq K$) iff the entropy functional $\mathcal{U}_{\infty,\mathfrak{m}}$ is $(K$-)convex along geodesics in $(\mathcal{P}(X), W_2)$. i.e. for every $\mu_0, \mu_1 \in \mathcal{P}(X)$ there exists a W_2-geodesic $(\mu_t)_{t\in[0,1]}$ such that for every $t \in [0,1]$ it holds

$$\mathcal{U}_{\infty,\mathfrak{m}}(\mu_t) \leq (1-t)\mathcal{U}_{\infty,\mathfrak{m}}(\mu_0) + t\mathcal{U}_{\infty,\mathfrak{m}}(\mu_1) - \frac{K}{2} t(1-t) W_2(\mu_0, \mu_1)^2.$$

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- DEF of $CD(K, N)$ condition [Lott-Sturm-Villani '06]: fixed $N \in [1, +\infty]$ and $K \in \mathbb{R}$, $(X, \mathsf{d}, \mathfrak{m})$ is a $CD(K, N)$-space if the Entropy $\mathcal{U}_{N,\mathfrak{m}}$ is K-convex along geodesics in $(\mathcal{P}(X), W_2)$ (for finite N is a "distorted" (K, N)-geod. conv.).

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- There are examples of Finsler manifolds which are CD spaces, e.g. $(\mathbb{R}^n, \|\cdot\|, \lambda^n)$ is $CD(0,n)$ for any norm $\|\cdot\|$.

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- There are examples of Finsler manifolds which are CD spaces, e.g. $(\mathbb{R}^n, \|\cdot\|, \lambda^n)$ is $CD(0,n)$ for any norm $\|\cdot\|$.
  $\rightsquigarrow$ $CD(K,N)$ spaces roughly are "possibly non-smooth Finsler manifolds with Ricci $\geq K$ and dimension $\leq N$"

Non completely satisfactory feature of $CD(K, N)$

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- Moreover, and maybe more importantly, some fundamental theorems in comparison geometry of Riemannian manifolds (e.g. Cheeger-Gromoll Splitting Theorem) are not true in the larger Finsler category (e.g. $(\mathbb{R}^2, \|\cdot\|_\infty)$ is $CD(0, 2)$, contains a line but does not split isometrically).

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- $\rightsquigarrow$ We would like to reinforce the $CD(K,N)$ condition in order to isolate the "Riemannian" $CD(K,N)$ spaces; in other words, we wish to rule out Finsler structures, but in a sufficiently weak way in order to still get a STABLE notion under mGH convergence.

Cheeger energy and $RCD(K, N)$ spaces

- Given a m.m.s. $(X, \mathsf{d}, \mathfrak{m})$ and $f \in L^2(X, \mathfrak{m})$, define the Cheeger energy

$$Ch_{\mathfrak{m}}(f) := \frac{1}{2} \int_X |\nabla f|_w^2 \, d\mathfrak{m} = \liminf_{u \to f \ in L^2} \frac{1}{2} \int_X (\mathrm{lip} u)^2 d\mathfrak{m}$$

where $|\nabla f|_w$ is the minimal weak upper gradient.

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Definition

Given $K \in \mathbb{R}$ and $N \in [1, \infty]$, $(X, \mathrm{d}, \mathfrak{m})$ is an $RCD(K, N)$ space if it is a $CD(K, N)$ space & the Cheeger energy is quadratic.

Cheeger energy and $RCD(K, N)$ spaces

- Given a m.m.s. $(X, \mathsf{d}, \mathfrak{m})$ and $f \in L^2(X, \mathfrak{m})$, define the Cheeger energy

$$Ch_{\mathfrak{m}}(f) := \frac{1}{2}\int_X |\nabla f|_w^2 \, d\mathfrak{m} = \liminf_{u \to f \; in L^2} \frac{1}{2}\int_X (\mathrm{lip} u)^2 d\mathfrak{m}$$

  where $|\nabla f|_w$ is the minimal weak upper gradient.
- Crucial observation: On a Finsler manifold M, the Cheeger energy is quadratic (i.e. parallelogram identity holds) iff M is Riemannian.
- Idea(Ambrosio-Gigli-Savaré): Reinforce the CD condition by asking that the Cheeger energy is quadratic.

Definition

Given $K \in \mathbb{R}$ and $N \in [1, \infty]$, $(X, \mathsf{d}, \mathfrak{m})$ is an $RCD(K, N)$ space if it is a $CD(K, N)$ space & the Cheeger energy is quadratic.

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- Weighted Riemannian manifolds with Bakry-Émery $N - Ricci \geq K$: i.e. (M^n, g) Riemannian manifold, let $\mathfrak{m} := \Psi \, vol_g$ for some smooth function $\Psi \geq 0$, then $Ric_{g,\Psi,N} := Ric_g - (N-n)\frac{\nabla^2 \Psi^{1/N-n}}{\Psi^{1/N-n}} \geq Kg$ iff $(M, \mathrm{d}_g, \mathfrak{m})$ is $RCD(K, N)$.

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- Cones or spherical suspensions over $RCD(N-1, N)$spaces (Ketterer)
- Quotients, orbifolds, metric-measure foliations with Ricci bounded below (GalazGarcia-Kell-M.-Sosa).
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A geometric application: Quotients by isometric group actions and lower Ricci bounds

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Q: If (M, g) has $Ric_g \geq Kg$, is the same true for the quotient space?

A special case

- If G acts freely (i.e no fixed points), then M^* is a smooth manifold and d^* is induced by a Riemannian metric g^*. Moreover $\mathrm{p} : (M, g) \to (M^*, g^*)$ is a Riemannian submersion.

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- Lott: YES!
- Q: what about the general case when the quotient space is not smooth?

Our result

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RK: previous work by Lott-Villani proving that $(M^*, \mathrm{d}^*, \mathfrak{m}^*)$ is $CD(K, \infty)$ or, in case $K = 0$, is $CD(0, N)$ under the assumption that M is compact. Apart from removing the compactness assumption and considering an arbitrary lower bound K, the geometric new content is that the quotient is infinitesimally hilbertian.

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$\rightarrow$ $RCD(K, N)$ spaces can be seen as an extension of the class of smooth Riemannian manifolds with Ricci $\geq K$, which is closed under many natural geometric and analytic operations.
Next lecture we will see some smooth appications.