# Some smooth applications of non-smooth Ricci curvature lower bounds 2<sup>nd</sup> Part: Smooth applications

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- ▶  $X = \mathbb{S}^n$  analogous: For all  $E \subset S^n$  it holds  $|\partial E| \ge |\partial B|$  where B is a metric ball (i.e. a spherical cap) s.t. |B| = |E|

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RK: In both of the examples the space is fixed (Euclidean space of Sphere), such a space contains a model subset (metric ball), and any subset of the space is compared with such a model subset.

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(2) (LGI) is global in the space, i.e. it does not depend just on E but also on  $M \setminus E$ : if one changes the space locally outside of E, the lhs in (LGI) may change since |M| may change E

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  - 2)  $E \simeq B$  metric ball.
- Question: Stability? i.e. If "=" in (LGI) is almost attained, Q1) What can we say on (M<sup>n</sup>, g)? Is it close to a sphere? In which sense?
  - Q2) What can we say on *E*? Is it close to a metric ball? In which sense?

## About Question Q1

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THM 1 (Particular case of Berard-Besson-Gallot, Inv. Math, 1985) Given  $(M^n,g)$  with  $Ric_g \geq (n-1)g$  and diam(M)=D (recall from Bonnet-Myers  $D \in (0,\pi)$ ) then

$$\frac{\mathcal{I}_{(M,g)}(v)}{\mathcal{I}_{\mathbb{S}^n}(v)} \ge \left(\frac{\int_0^{\pi/2} (\cos t)^{n-1} dt}{\int_0^{D/2} (\cos t)^{n-1} dt}\right)^{1/n}$$

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#### RK:

- 1) rhs is  $\geq 1$  so the result sharpens the classical LGI
- 2) It follows that there exists  $C_{n,v}>0$  such that if for some  $v\in(0,1)$  it holds  $\mathcal{I}_{(M,g)}(v)\leq\mathcal{I}_{\mathbb{S}^n}(v)+\delta$ , then

$$\pi - D \leq C_{n,v} \delta^{1/n}$$
.

## Answering Question 2 in Euclidean setting

#### Quantitative Euclidean Isoperimetric Inequality

(Fusco-Maggi-Pratelli, Annals of Math. 2008) There exists  $C_n>0$  such that for every  $E\subset\mathbb{R}^n$  there exists a round ball  $B\subset\mathbb{R}^n$  with |E|=|B| and

$$\frac{|E\Delta B|}{|E|} \le C_n \left(\frac{|\partial E|}{|\partial B|} - 1\right)^{1/2}$$

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RK: 1) the rhs is the so-called "isoperimetric deficit" and is zero iff E is a ball (by rigidity in EII).

- 2) The proof of FMP is via a "quantitative symmetrization".
- 3) Alternative proof of the result via Brenier  $L^2$ -Optimal Transport map (by Figalli-Maggi-Pratelli, Inv. Math. 2010) and via regularity theory and selection principle (Cicalese-Leonardi, ARMA 2012).

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#### Quantitative Spherical Isoperimetric Inequality

(Bogelein-Duzaar-Fusco, Adv. Calc. Var. 2015)

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Proof: along the same lines of Cicalese-Leonardi's selection principle.

## Difficulties about Question 2: quantitative Levy-Gromov inequality

The above quantitative isoperimetric inequalities are for a fixed space ( $\mathbb{R}^n$  or  $\mathbb{S}^n$ ), with the highest possible degree of symmetry. LGI is for any  $(M^n,g)$  with  $Ric_g \geq (n-1)g$ 

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  - ► Symmetrization (FMP): since *M* is not symmetric it makes little sense to speak of symmetrization.
  - ▶ Brenier Map,  $L^2$ -OT (FMP): works in  $\mathbb{R}^n$  but already in  $\mathbb{S}^n$  it is an open problem to prove Spherical Isoperimetric Inequality via Brenier Map.
  - Selection Principle (CL): would need smooth convergence of metrics while here the natural convergence is Gromov-Hausdorff.

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## The result: quantitative Levy-Gromov inequality

THM 2 (Cavalletti-Maggi-M. in press in CPAM)

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In particular, if  $E \subset M$  is an isoperimetric subset with  $\frac{|E|}{|M|} = v$ , then

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RK Difference with QEII or QSII: here  $E\subset M$  and  $|\partial E|$  is compared with  $\mathcal{I}_{\mathbb{S}^n}$  (not of  $\mathcal{I}_{(M,g)}$ ) via a "Levy-Gromov isoperimetric deficit".

## The result holds in higher generality

Actually we prove THM1 and THM 2 more generally for RCD(N-1,N) metric measure spaces spaces. As we saw in the first lecture, examples entering this class of spaces:

- ▶ mGH-limits of Riemannian *N*-dimensional manifolds satisfying  $Ric_g \ge (N-1)g$ .
- ▶ *N*-dimensional Alexandrov spaces with curvature  $\geq 1$ .

Part 2. Ricci flow, Perelman's Pseudo Locality Theorem and Almost euclidean isoperimetric inequalities.

## Perelman's Pseudo-locality Theorem

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$$R_{g_0}(x) \ge -1$$
 &  $|\partial \Omega|_{g_0} \ge (1-\delta) c_n |\Omega|_{g_0}^{(n-1)/n}, \ \forall x, \Omega \subset B_1(x_0),$ 

where  $c_n$  is the euclidean isoperimetric constant.

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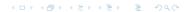
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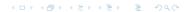
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$$Ric_{g_0}(x) \ge -\delta^2 g_0 \& |B|_{g_0} \ge (1-\delta) \omega_n \stackrel{?}{\Rightarrow} |\partial\Omega|_{g_0}^n \ge (1-\varepsilon) c_n |\Omega|_{g_0}^{n-1}$$
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THM[Cavalletti-M. IMRN '18] For every  $N \in [2,\infty) \cap \mathbb{N}$  there exist  $\bar{\varepsilon}_N, \bar{\delta}_N, C_N > 0$  such that the next holds. Let (M,g) be a smooth N-dim. Riem. manifold and let  $\bar{x} \in M$ . Assume that  $B_1(\bar{x})$  is rel. compact and for some  $\delta \in [0,\bar{\delta}_N]$   $Ric_{\sigma} \geq -\delta^2 g$  on  $B_1(\bar{x})$  &  $|B_1(\bar{x})| \geq (1-\delta)\omega_N$ 

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RK Actually we prove the corresponding statement more generally for a m. m. space  $(X, d, \mathfrak{m})$  which is essentially non-branching,  $CD_{loc}(-\delta, N)$  on a ball  $B_1(\bar{x})$  and  $\mathfrak{m}(B_1(\bar{x})) \geq (1 - \delta)\omega_N$ .

COR[Cavalletti-M. IMRN '18] For every  $N \in [2, \infty) \cap \mathbb{N}$  there exist  $\bar{\varepsilon}_N, \bar{\delta}_N, C_N > 0$  such that the next holds.

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RK closeness in GH-distance is a priori a very weak assumption (a manifold is  $\delta$ -GH close to a  $\delta$ -net which is a discrete space); so it is remarkable that GH-close + lower Ricci bound  $\Rightarrow$  almost euclidean isoperimetric inequality.



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Classical method for proving Levy-Gromov isoperimetric inequality in a nutshell:

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# Part 3. Some ideas of the 1D localization technique

Let  $(X, d, \mathfrak{m})$  be a CD(K, N) space and assume for the moment that given  $E \subset X$  we can find a "1-D localization"  $\{X_q\}_{q \in Q}$  of X, i.e.

1.  $\{X_q\}_{q\in Q}$  is (essentially) a partition of X, i.e.  $\mathfrak{m}(X\setminus \mathring{\bigcup}_{q\in Q}X_q)=0$ ,

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- 2.  $\mathfrak{m}=\int_{Q}\mathfrak{m}_{q}\,\alpha(dq)$ , with  $\alpha(Q)=1$  and  $\mathfrak{m}_{q}(X_{q})=\mathfrak{m}_{q}(X)=1$  for  $\alpha$ -a.e.  $q\in Q$   $\leadsto$  disintegration of  $\mathfrak{m}$  (kind of non-straight Fubini)

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RK the first two assumptions are mild, the characterizing properties are the last two.



$$\mathfrak{m}^{+}(E) := \liminf_{\varepsilon \to 0^{+}} \frac{\mathfrak{m}(E^{\varepsilon}) - \mathfrak{m}(E)}{\varepsilon}$$

$$= \liminf_{\varepsilon \to 0^{+}} \int_{Q} \frac{\mathfrak{m}_{q}(E^{\varepsilon}) - \mathfrak{m}_{q}(E)}{\varepsilon} \alpha(dq) \text{ by 2.}$$

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- Consider the L<sup>1</sup>-optimal transport problem

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▶ By Optimal Transport techniques there exists a minimizer  $\gamma \in \mathcal{P}(X \times X)$  and a 1-Lipschitz function  $\varphi : X \to \mathbb{R}$  called Kantorovich potential such that, denoted

$$\Gamma := \{(x,y) \in X \times X : \varphi(x) - \varphi(y) = \mathsf{d}(x,y)\},\$$

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- ▶ More work to prove properties 3. and 4.



It is more standard to consider the  $L^2$ -optimal transport problem: given  $\mu_0, \mu_1 \in \mathcal{P}(X)$  let

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- Extension to non-smooth spaces by Cavalletti-M. '15.



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