Some smooth applications of non-smooth Ricci curvature lower bounds
2^{nd} Part: Smooth applications

Andrea Mondino (University of Warwick)

Workshop in Geometric Analysis
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Plan of the talk

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- $X = \mathbb{R}^n \rightsquigarrow$ Euclidean isoperimetric inequality:
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RK: In both of the examples the space is fixed (Euclidean space of Sphere), such a space contains a model subset (metric ball), and any subset of the space is compared with such a model subset.

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Besides the euclidean one, probably the most famous isoperimetric inequality is the Levy-Gromov isoperimetric inequality:

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(2) (LGI) is global in the space, i.e. it does not depend just on E but also on $M \setminus E$: if one changes the space locally outside of E, the lhs in (LGI) may change since $|M|$ may change.

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- Levy-Gromov Inequality can be stated as: Given (M^n, g) with $Ric_g \geq (n-1)g$ then

$$\mathcal{I}_{(M,g)}(v) \geq \mathcal{I}_{\mathbb{S}^n}(v), \; \forall v \in [0,1].$$

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  1) $(M^n, g) \simeq \mathbb{S}^n$ isometric
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- Question: Stability? i.e. If "=" in (LGI) is almost attained,
  Q1) What can we say on (M^n, g)? Is it close to a sphere? In which sense?
  Q2) What can we say on E? Is it close to a metric ball? In which sense?

About Question Q1

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THM 1 (Particular case of Berard-Besson-Gallot, Inv. Math, 1985) Given (M^n, g) with $Ric_g \geq (n-1)g$ and $\text{diam}(M) = D$ (recall from Bonnet-Myers $D \in (0, \pi)$) then

$$\frac{\mathcal{I}_{(M,g)}(v)}{\mathcal{I}_{\mathbb{S}^n}(v)} \geq \left(\frac{\int_0^{\pi/2} (\cos t)^{n-1} dt}{\int_0^{D/2} (\cos t)^{n-1} dt} \right)^{1/n}$$

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RK:

1) rhs is ≥ 1 so the result sharpens the classical LGI

2) It follows that there exists $C_{n,v} > 0$ such that if for some $v \in (0,1)$ it holds $\mathcal{I}_{(M,g)}(v) \leq \mathcal{I}_{\mathbb{S}^n}(v) + \delta$, then

$$\pi - D \leq C_{n,v} \delta^{1/n}.$$

Answering Question 2 in Euclidean setting

Quantitative Euclidean Isoperimetric Inequality
(Fusco-Maggi-Pratelli, Annals of Math. 2008)
There exists $C_n > 0$ such that for every $E \subset \mathbb{R}^n$ there exists a round ball $B \subset \mathbb{R}^n$ with $|E| = |B|$ and

$$\frac{|E \Delta B|}{|E|} \leq C_n \left(\frac{|\partial E|}{|\partial B|} - 1 \right)^{1/2}$$

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RK: 1) the rhs is the so-called "isoperimetric deficit" and is zero iff E is a ball (by rigidity in EII).
2) The proof of FMP is via a "quantitative symmetrization".
3) Alternative proof of the result via Brenier L^2-Optimal Transport map (by Figalli-Maggi-Pratelli, Inv. Math. 2010) and via regularity theory and selection principle (Cicalese-Leonardi, ARMA 2012).

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$$|E \Delta B| \leq C_{n,v} \left(\frac{|\partial E|}{|\mathbb{S}^n|} - \mathcal{I}_{\mathbb{S}^n}(v) \right)^{1/2}$$

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Proof: along the same lines of Cicalese-Leonardi's selection principle.

Difficulties about Question 2: quantitative Levy-Gromov inequality

The above quantitative isoperimetric inequalities are for a fixed space ($\mathbb{R}^n$ or $\mathbb{S}^n$), with the highest possible degree of symmetry.

LGI is for any (M^n, g) with $Ric_g \geq (n-1)g$

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- Symmetrization (FMP): since M is not symmetric it makes little sense to speak of symmetrization.
- Brenier Map, L^2-OT (FMP): works in $\mathbb{R}^n$ but already in $\mathbb{S}^n$ it is an open problem to prove Spherical Isoperimetric Inequality via Brenier Map.
- Selection Principle (CL): would need smooth convergence of metrics while here the natural convergence is Gromov-Hausdorff.

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$\rightsquigarrow$ Novel approach: localization via L^1-Optimal Transport.

The result: quantitative Levy-Gromov inequality

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For every $v \in (0,1)$ and $n \geq 2$ there exists $C_{n,v} > 0$ with the following properties.
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In particular, if $E \subset M$ is an isoperimetric subset with $\frac{|E|}{|M|} = v$, then

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RK Difference with QEII or QSII: here $E \subset M$ and $|\partial E|$ is compared with $\mathcal{I}_{\mathbb{S}^n}$ (not of $\mathcal{I}_{(M,g)}$) via a "Levy-Gromov isoperimetric deficit".

The result holds in higher generality

Actually we prove THM1 and THM 2 more generally for $RCD(N-1, N)$ metric measure spaces spaces. As we saw in the first lecture, examples entering this class of spaces:

- mGH-limits of Riemannian N-dimensional manifolds satisfying $Ric_g \geq (N-1)g$.
- N-dimensional Alexandrov spaces with curvature ≥ 1.

Part 2. Ricci flow, Perelman's Pseudo Locality Theorem and Almost euclidean isoperimetric inequalities.

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$$R_{g_0}(x) \geq -1 \quad \& \quad |\partial\Omega|_{g_0} \geq (1-\delta)\, c_n |\Omega|_{g_0}^{(n-1)/n}, \ \forall x, \Omega \subset B_1(x_0),$$

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Let (M,g) be a smooth N-dim. Riem. manifold and let $\bar{x} \in M$. Assume that $B_1(\bar{x})$ is rel. compact and for some $\delta \in [0,\bar{\delta}_N]$

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Let (M,g) be a smooth N-dim. Riem. manifold and let $\bar{x} \in M$. Assume that $B_1(\bar{x})$ is rel. compact and for some $\delta \in [0, \bar{\delta}_N]$

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Almost euclidean isoperimetric inequality

Q: do the assumptions of Tian-Wang's Pseudo-locality imply the assumptions of Perelman's Pseudo-Locality? I.E.

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RK Actually we prove the corresponding statement more generally for a m. m. space $(X, \mathsf{d}, \mathfrak{m})$ which is essentially non-branching, $CD_{loc}(-\delta, N)$ on a ball $B_1(\bar x)$ and $\mathfrak{m}(B_1(\bar x)) \geq (1-\delta)\omega_N$.

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RK closeness in GH-distance is a priori a very weak assumption (a manifold is δ-GH close to a δ-net which is a discrete space); so it is remarkable that GH-close + lower Ricci bound $\Rightarrow$ almost euclidean isoperimetric inequality.

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DIFFICULTY If we want to prove an AE isoperimetric ineq on $B_1(\bar{x})$

- A minimizing sequence for the perimeter can approach $\partial B_1(\bar{x})$ and so the minimizer Ω will hit $\partial B_1(\bar{x})$.

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Part 3. Some ideas of the 1D localization technique

Proof part 1: 1-D localization

Let $(X, \mathrm{d}, \mathfrak{m})$ be a $CD(K, N)$ space and assume for the moment that given $E \subset X$ we can find a "1-D localization" $\{X_q\}_{q \in Q}$ of X, i.e.

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RK the first two assumptions are mild, the characterizing properties are the last two.

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If for a given $E \subset X$ we can find a 1-D localization as above then

$$\begin{aligned} \mathfrak{m}^+(E) &:= \liminf_{\varepsilon \to 0^+} \frac{\mathfrak{m}(E^\varepsilon) - \mathfrak{m}(E)}{\varepsilon} \\ &= \liminf_{\varepsilon \to 0^+} \int_Q \frac{\mathfrak{m}_q(E^\varepsilon) - \mathfrak{m}_q(E)}{\varepsilon} \, \alpha(dq) \quad \text{by 2.} \end{aligned}$$

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- More work to prove properties 3. and 4.

Why L^1-trasport?

- It is more standard to consider the L^2-optimal transport problem: given $\mu_0, \mu_1 \in \mathcal{P}(X)$ let

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- On the other hand L^1 transport does induce an equivalence relation into rays where the transport is performed
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- Extension to non-smooth spaces by Cavalletti-M. '15.

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