Local Ricci flow and limits of non-collapsed regions with Ricci curvature bounded below

Miles Simon
(University of Magdeburg, Germany)
Joint work with Peter Topping

Survey Talks,
Paris (IHP), 5 June, 2018

This work, unless stated otherwise, is joint work with Peter Topping.

The Ricci Flow equation

M^n smooth manifold, $dim(M^n) = n$.

We reserve the word *manifold* for *connected manifolds* (unless otherwise stated).

$\partial M^n = \emptyset$

$(M^n, g(t))_{t \in I}$ solves Ricci-Flow (RF) if g is smooth in space and time and solves

$$\frac{\partial}{\partial t} g(x,t) = -2\mathrm{Rc}(g(\cdot,t))(x)$$

for all $x \in M$ for all $t \in I$.

I is an interval.

$\mathrm{Rc}(g) =$ the Ricci curvature of g, (a $(0,2)$ tensor)

$\mathrm{R}(g) =$ scalar curvature of g, (a function)

$\mathrm{Rm}(g)$ is the full curvature tensor (a $(0,4)$ tensor).

In the theorems and discussions that follow it could be that $(M^n, g(t))$ is not complete and/or not compact , and/or does not have bounded curvature for some (or all) $t \in I$. If the solution is complete then we call it a (CMPL) solution, if it has bounded curvature, that is $\sup_{M \times I} |\mathrm{Rm}(g)| < \infty$, then we call it a (BC) solution.

Examples:

Theorem 1 (Ha. '82, Shi '89,)

If (M, g_0) is smooth, complete (CMPL) and has bounded curvature (BC),
$k_0 := \sup_M |\mathrm{Rm}(g_0)| < \infty$*, then there exists a unique (B.-L. Chen, X.-P. Zhu '06) smooth, solution* $(M, g(t))_{t\in[0,T(n,k_0)]}$ *to Ricci flow which is complete for all* $t \in [0, T]$ *and has* $\sup_{M\times[0,T]} |\mathrm{Rm}(g)| < \infty$*. We call a complete solution to Ricci flow which has bounded curvature in this sense, a (CMPL) (BC) solution. In fact, w.l.o.g., the construction guarantees that* $\sup_{M\times[0,T]} |\mathrm{Rm}(g)| \leq 2k_0$ *(doubling estimate).*

Local properties of Ricci Flow

(A simplified version of) Perelman's Pseudolocality Theorem vn. 2 says:

If $(M, g(t))_{t\in[0,T]}$ is (BC) and (CMPL), and **and**

$$|\mathrm{Rm}(g_0)| \le 1 \text{ on } B_{g_0}(x_0, 1) \text{ and}$$
$$\mathrm{vol}(B_{g_0}(x_0, 1) \ge v_0,$$

then there exists a $S(v_0, n) > 0$ such that,

$$|\mathrm{Rm}(g(t))| \le c_0(n) \text{ on } B_{g_0}(x_0, \frac{1}{2})$$

for all $t \le S(v_0, n)$.

Local Poperty of RF: Compact subsets of time zero regular open regions are regular at later small times (*small* can be quantified). , if the solution is (BC) and (CMPL) .

Regularising effect of RF (A simplified version of) Perelman's Pseudolocality Theorem vn. 1 says: If $(M^n, g(t))_{t\in[0,T]}$ is (BC) and (CMPL), and $\mathrm{R}(g_0) \geq -1$ on $B_{g_0}(x_0, 1) \subseteq M$ **and** there are coordinates $\varphi : B_{g_0}(x_0, 1) \to \varphi(B_{g_0}(x_0, 1)) \subseteq \mathbb{R}^n$ such that (in these coordinates) $(1-\epsilon)\delta \leq g_0 \leq (1+\epsilon)\delta$, where $\epsilon \leq \epsilon_0(n)$ is small enough, then there exists $S(n) > 0$ such that

$$|\mathrm{Rm}(g(t))| \leq \frac{\alpha(\epsilon)}{t} \text{ on } B_{g_0}(x_0, \frac{1}{2}).$$

for all $t \in [0, S] \cap [0, T]$, where $\alpha(\epsilon) \to 0$ with $\epsilon \to 0$.

Regularising effect of RF Compact subsets of *Rough* open regions are smoothed out by (RF) for small (quantifiable) times if the solution is (BC) and (CMPL).
eg. $n = 2$ $\frac{\partial}{\partial t}\mathrm{R} = \Delta_g \mathrm{R} + \mathrm{R}^2$. Reaction-Diffusion equation.

Pictures $+$ comparison with heat flow.

Does the regularising effect smooth out steep cones (for $n = 2, 3$)? Yes, if they are almost positively curved: Global Regularising effect in dimension $n = 2, 3$.

Theorem 2 (Si. ,'11 : Global regularising effect)

Let (M^3, g_0) *be smooth, and satisfy (BC), (CMPL) and*
a) (global non-collapsedness) $\mathrm{vol}(B_{g(0)}(x,1)) \geq v_0$ *for all* $x \in M$ *and*
b) $\mathrm{Rc}(\cdot, 0) \geq -1.$ *Then the* **maximal** *Shi /Hamilton solution to RF exists for at least* $t \in [0, T(v_0)]$ *and has* $|\mathrm{Rm}|(\cdot, t) \leq \frac{c_0(v_0)}{t}$, *and* $\mathrm{Rc}(\cdot, t) \geq -K(v_0)$ *for* $t \in [0, T(v_0)]$. *Furthermore the following distance estimates are satisfied:*
$e^t d_0(x,y) \geq d_t(x,y) \geq d_0(x,y) - \gamma(3)\sqrt{c_0 t}$,
for all $x, y \in M$, $t \geq s \in [0, T(v_0)]$, *where* $d_t(x,y) = dist(g(t))(x,y)$.

An application: consider a pointed sequence $(M_i^3, g_i(0), p_i)$ of initial (BC), (CMPL) Riemannian manifolds satisfying these conditions, that is
a) (global non-collapsedness) $\mathrm{vol}(B_{g_i(0)}(x,1)) \geq v_0$ for all $x \in M_i$ and
b) $\mathrm{Rc}(g_i(0))(\cdot) \geq -1.$ Using the existence theorem and the estimates contained in the statement of the theorem, and the smooth convergence theorem of Cheeger/Hamilton , we may let i go to infinity, and we obtain (after taking a subsequence) a smooth limiting solution defined for $t \in (0,(T(v_0))$:
$(M^3, g(t), p) := \lim_{i\to\infty}(M_i, g_i(t), p_i)_{t\in(0,T(v_0))}$, the limit being in the smooth Cheeger/Hamilton sense, and the estimates carry over to the limit.

$(M^3, g(t), p) := \lim_{i\to\infty}(M_i, g_i(t), p_i)_{t\in(0,T(v_0))}$, with $|\mathrm{Rm}(\cdot,t)| \leq \frac{c_0(v_0)}{t}$ and the distance estimates hold. The distance estimates guarantee that there is a limit $(X, d_X, x_\infty) = \lim_{t\searrow 0}(M^3, d(g(t)), p)$, with the same topolgy as X, that is X is a manifold. The compactness theorem of Gromov guarantees that (after taking a subsequence) there is a Gromov-Hausdorff Limit $(Z, d_Z, z) := \mathrm{GHlim}_{i\to\infty}(M_i, d(g_i(0)), p_i)$. The distance estimates, once again guarnatee that, $(X, d_X, x_\infty) = (Z, d_Z, z)$. That is Z must be a topological manifold. This proves a weak version of the conjecture of M. Anderson, J. Cheeger, T. Colding, G. Tian. Weak because we assume global non-collapsedness and BC.

In recent works with Peter Topping, we show that the full conjecture of Anderson, Cheeger, Colding, Tian (ACCT Conjecture) is correct. The full conjecture removes the condition (BC) and replaces the global non-collapsedness condition by a non collapsed in a point condition:

Theorem 3 (Si.+ Topping, '17)

Let $(M_i^3, g_i(0))$ *be a sequence of complete (CMPL)* 3*-manifolds with*
a) (non-collapsed in a point) $\mathrm{vol}(B_{g_i(0)}(x_i, 1)) \geq v_0$ *for some* $x_i \in M_i$ *and*
b) $\mathrm{Rc}(g_i(0)) \geq -1$.
Then a subsequence of $(M, d(g_i(0)), x_i) \to (X, d_X, x_\infty)$ *in the Gromov-Hausdorff sense when* $i \to \infty$, *where* (X, d_X, x_∞) *is a metric space, and* X *with the topology induced by* d_X, *is a topological* 3*-manifold.*

The proof of the full ACCT conjecture involves
(1) localising all of the Ricci flow estimates from Si. '11 and
(2) constructing a local Ricci flow $(B_{g_i(0)}(x_0, R), g_i(t))_{t\in[0,T(R,v_0)]}$ for each Ball $B_{g_i(0)}(x_0, R) \subseteq (M_i, g_i(0))$ and taking a limit as $i \to \infty$ of each of these flows.
We rely on some earlier local estimates (Si.+ Topping '16) and a construction of R. Hochard ('16).Local Ricci flows were first considered by Hochard, ('16).

We only consider here the case of balls of radius $R = 4$. The local Ricci flow result we obtain, is an existence result with estimates

Other results and open problems:
Shortly after the release of this paper, ('17), M.-C. Lee and L.-F. Tam, proved an analogous result, using the methods of Simon/Topping, estimates from other papers (M.-C. Lee, L.-F. Tam, A. Chau, C. Yu, S. Huang, ...), and new ideas/estimates, in the case that (M^{2m}, g_0) is globally non-collapsed, Kähler and has non-negative bisectional curvature ($\mathrm{Rm}(\cdot)(X, \bar{X}, Y, \bar{Y}) \geq 0$ for any $X, Y \in T^{1,0}_{(\cdot)}M$).
They show that a Kähler Ricci flow solution $(M, g(t))_{t \in [0, T(v_0, n))}$ exists and has non-negative bi-sectional curvature and $|\mathrm{Rm}(\cdot, t)| \leq \frac{c_0(v_0, n)}{t}$ for $t \in [0, T(v_0, n))$.

R. Bamler, B. Wilking, E. Cabezas-Rivas ('17) showed using different methods (heat kernel estimates and integral arguments), that analogous results may be obtained for other curvature quantities for all dimensions. In the following, $\mathrm{I}(g)$ will refer to the curvature operator of the unit sphere : $\mathrm{I}(g)_{ijkl} = g_{ik}g_{jl} - g_{il}g_{jk}$

Their assumptions are that g_0 is globally non-collapsed, and $\mathrm{Rm}(g_0) + \mathrm{I}(g_0)$ is a (1) non-negative curvature operator,(2) 2-non-negative curvature operator and g_0 is (BC), (3) non-negative complex sectional curvature operator, (4) a weakly PIC_1 curvature operator and g_0 is (BC) (5) is a non-negative bisectional curvature operator and g_0 is (BC) and Kähler w.r.t. to some complex struture J. They show that a solution $(M, g(t))_{t\in[0,T(v_0,n))}$ exists with $\mathrm{Rm}(g(t)) + c_0(v_0, n)\mathrm{I}(g(t))$ is a non-negative curvature operator in case (1), etc, and one has $|\mathrm{Rm}(\cdot, t)| \leq \frac{c_0(v_0,n)}{t}$.

They do not construct local Ricci flows in their proofs, hence the assumption (BC) in (2),(4),(5). They achieved (1) and (3) without (BC), by making a (specially chosen) conformal change, which keeps the metric fixed on balls of radius R, and results in a (BC) space whose curvature bound from below, and the volume of balls of radius one, only changes by a fixed (independent of R) constant factor.

In a recent paper by Yi Lai ('18), the estimates and methods of Bamler/Wilking/Cabezas-Rivas and the methods of Simon/Topping, as well as new methods and ideas, are used , and she was able to (a) remove the (BC) condition from (2) and (4), and
(b) prove that the limiting spaces are manifolds, if the global volume non-collapsing is replaced by volume non-collapsed in a point.

That is assuming $\mathrm{vol}(B_{g(0)}(x_0, 1)) = v_0$ and $B_{g(0)}(x_0, 1)$ satisfies (1),(2),(3),(4), Yi Lai constructs a local Ricci flow solution $(B_{g(0)}(x_0, 1), g(t))_{t \in [0, T(v_0, n)]}$ with $|\mathrm{Rm}(\cdot, t)| \leq \frac{c_0(v_0, n)}{t}$ and $\mathrm{Rm}(g(t)) + c_0(v_0, n)\mathrm{I}(g(t))$ is a non-negative curvature operator in case (1), etc.

Open problem Can one construct a local Ricci flow of this type in case (5) (Kähler and bi-sectional curvature not less than minus one), with good estimates, as in case (1)-(4).

Theorem 4 (Si./Topping '17: The Local Flow Theorem)

Let (M^3, g_0) *be smooth,* $x_0 \in M$, $B_{g_0}(x_0, 4) \subset\subset M$, *such that*

$$\mathrm{Rc}(g_0) \geq -1 \text{ on } B_{g_0}(x_0, 4)$$
$$\mathrm{vol}(B_{g_0}(x_0, 4)) \geq v_0.$$

Then there exists a $S = S(v_0, n)$, $c_0(v_0, n)$ *and a solution* $(B_{g_0}(x_0, 1), g(t))_{t \in [0,S]}$ *to (RF) such that*

$$\mathrm{Rc}(g(t)) \geq -c_0$$
$$|\mathrm{Rm}(g(t))| \leq \frac{c_0}{t}$$

for all $t \in [0, S]$.

R. Hochard '16 proved a similar result, obtaining $\mathrm{Rc}(g(t)) \geq -\frac{1}{t}$ in place of $\mathrm{Rc}(g(t)) \geq -c_0$.

Using the estimates + Lemmata in this talk + Pseudolocality estimate (vn. 1) of Perelman, one can show a local Pseudolocality type Theorem in a Ricci setting for $n = 3$.

Theorem 5 (Si./Topping '17)

Let $(M^3, g(t))_{t\in[0,T]}$ *be a smooth (BC), (CMPL) solution to (RF),* $x_0 \in M$*, and*

$$\mathrm{Rc}(g_0) \geq -1 \text{ on } B_{g_0}(x_0, 4)$$
$$\mathrm{vol}(B_{g_0}(x_0, 4)) \geq v_0.$$

Then there exists a $S = S(v_0, n), c_0(v_0, n)$ *such that*

$$\mathrm{Rc}(g(t)) \geq -c_0$$
$$|\mathrm{Rm}(g(t))| \leq \frac{c_0}{t}$$

for all $t \in [0, S] \cap [0, T]$ *on* $B_{g(t)}(x_0, 1)$

Once again, R. Hochard '16 proved a similar result, obtaining $\mathrm{Rc}(g(t)) \geq -\frac{1}{t}$ in place of $\mathrm{Rc}(g(t)) \geq -c_0$.

Open Problem: Do local results of this type exist for the other curvature conditions mentioned?
Open Problem: Can one remove the (BC) condition?
Note: For $n = 2$ we can remove (BC) and (CMPL) if we include $B_{g(t)}(x_0, 4) \subset\subset M$ for all $t \in [0, T]$ (Si. '13).

The proof of the distance estimates of Si. '11 is local: if $|\mathrm{Rm}|(\cdot,t) \leq \frac{c_0}{t}$ and $\mathrm{Rc}(g(t)) \geq -1$ on $B_{g(t)}(1,x_0)$ and $B_{g(t)}(1,x_0) \subset\subset N$ for all $t \in [0,T)$, for a **not necessarily CMPL solution** $(N,g(t))_{t\in[0,T)}$ then

$$e^t d_0(x,y) \geq d_t(x,y) \geq d_0(x,y) - \gamma(n)\sqrt{c_0 t}$$

for all $x,y \in B_{g(t)}(x_0,\frac{1}{4})$, $t \in [0,T(c_0)]$. In '17 (joint work with Peter Topping), we proved the following improvement of this:

$$\begin{aligned} e^t d_0(x,y) &\geq d_t(x,y) \geq d_0(x,y) - \gamma(n)\sqrt{c_0 t}, \textit{ and} \\ d_t(x,y) &\geq \beta(n,c_0)[d_0(x,y)]^{1+2(n-1)c_0}. \end{aligned}$$

for all $x,y \in B_{g(t)}(x_0,\frac{1}{4})$, $t \in [0,T(c_0)]$.

Assuming $|\mathrm{Rm}(\cdot, t)| \leq \frac{c_0}{t}$ and $\mathrm{Rc}(g(t)) \geq -g(t)$ (everywhere), the distance estimates of Si./Topping '17 are shown as follows.
For a curve $\gamma : [0, r] \to M$, the length of γ with respect to $g(t)$ is
$L_{g(t)}(\gamma) := \int_0^r \sqrt{g(t)(\frac{d\gamma}{ds}(s), \frac{d\gamma}{ds}(s))} ds.$ Using $\frac{\partial}{\partial t} g(t) e^{-2t} = -2\mathrm{Rc}(g(t)) - 2g(t) \leq 0$, we see that $\frac{\partial}{\partial t}(e^{-t} L_{g(t)}(\gamma)) \leq 0$ and hence, integrating in time, $L_{g(t)}(\gamma) \leq e^t L_{g(0)}(\gamma)$ which implies $d_t(x, y) \leq e^t d_0(x, y).$

The other estimate is obtained as follows. At times t where the (Lipschitz in time) function $d_t(x,y) = L_{g(t)}(\gamma)$ is differentiable, there exists a length minimising geodesic γ (w.r.t to t) from x to y such that $\frac{d}{dt} d_t(x,y) \geq \frac{d}{dt} L_{g(t)}(\gamma) = \int_0^r -\mathrm{Rc}(g(t))(\frac{d\gamma}{ds}(s), \frac{d\gamma}{ds}(s))ds$ [Hamilton's Integral version of Myers' Theorem:]
$\geq -\gamma(n)\sqrt{\sup_M |\mathrm{Rm}(g(t)|} \geq -\gamma(n)\frac{\sqrt{c_0}}{\sqrt{t}}$.
Integrating in time from $t = 0$ to $t \leq t_1 := \frac{1}{c_0}[\frac{1}{2\gamma} d_0(x,y)]^2 < 1$ implies $d_t(x,y) \geq \frac{1}{2} d_0(x,y)$.

Now using $\mathrm{Rc} \leq \frac{(n-1)c_0}{t}$ for $t \geq t_1$ we get
$\frac{d}{dt} d_t(x,y) \geq \int_0^r -\mathrm{Rc}(g(t))(\frac{d\gamma}{ds}(s), \frac{d\gamma}{ds}(s))ds$
$\geq -d_t(x,y)\frac{(n-1)c_0}{t}$. Integrating from $t = t_1$ to $t \leq 1$ gives $d_t(x,y) \geq d_{t_1}(x,y)[\frac{t}{t_1}]^{-(n-1)c_0}$
$\geq d_{t_1}(x,y) t_1^{(n-1)c_0}$
$\geq \frac{1}{2} d_0(x,y) t_1^{(n-1)c_0} = \beta(c_0, n)[d_0(x,y)]^{1+2(n-1)c_0}$ as required.

We write down the three main local results that we need, to construct our local Ricci flow.

Lemma 6 (Local Lemma Si. + Topping '16)

Let $(N^3, g(t))_{t\in[0,T]}$ *be a smooth Ricci flow such that for some fixed* $x \in N$ *we have* $B_{g(t)}(x_0, 1) \subset\subset N$ *for all* $t \in [0, T]$*, and so that*
(i) $\mathrm{vol}(B_{g(0)}(x_0, 1)) \geq v_0 > 0$*, and*
(ii) $\mathrm{Rc}_{g(t)} \geq -1$ *on* $B_{g(t)}(x_0, 1)$ *for all* $t \in [0, T]$*.*
Then there exist $C_0 = C_0(v_0) \geq 1$ *and* $\hat{T} = \hat{T}(v_0) > 0$ *such that* $|\mathrm{Rm}|_{g(t)}(\cdot) \leq C_0/t$*, and* $\mathrm{inj}_{g(t)}(\cdot) \geq \sqrt{t/C_0}$ *on* $B_{g(t)}(x_0, \frac{1}{2})$ *for all* $0 < t \leq \min(\hat{T}, T)$.

The proof is almost the same as the proof given in Si. '11, which is, as closer examination shows, a local argument: Assuming the injectivity radius estimate, the curvature estimate is proved using an argument by contradiction: After scaling our contradiction sequence of solutions appropriately, we obtain as a limit a smooth, ancient (defined for $t \in (-\infty, 0]$), non-compact, smooth (BC, CMPL), non-trivial solution, with Euclidean volume growth and $\sec \geq 0$ everywhere (D. Knopf/B. Chow $n = 3$ and (BC), B.-L. Chen, $n = 3$). A Theorem of Perleman says such solutions don't exist.

Using the curvature estimate, we show that the injectivity radius estimate does indeed hold: A theorem of Cheeger+Colding ('97) [volume is continuous under convergence of metric spaces (Cheeger+ Colding, '97) , as long as Ricci curvature stays bounded from below]+ local distance estimates (as explained above) + curvature estimate imply the injectivity radius estimate.

Note that the results of Bamler/Wilking/Cabezas-Rivas, Lee/Tam..., all use a local lemma proved for the curvature condition they consider. The proofs are similar to the one just given, except that the theorem of Perelman is in each case replaced by an analogous theorem for the curvature condition being considered (for example in the bisectional curvature case, such a Theorem is provided by L. Ni).

The second lemma we need is:

Lemma 7 (cf. Double Bootstrap DB Lemma Si.+Topping '16)

Let $(N^3, g(t))_{t\in[0,T]}$ *be a smooth Ricci flow, and* $x_0 \in N$, *such that* $B_{g(0)}(x_0, 2)$ *is compactly contained in* N *and so that throughout* $B_{g(0)}(x_0, 2)$ *we have*

i) $|\mathrm{Rm}|_{g(t)} \leq \frac{c_0}{t}$ *for some* $c_0 \geq 1$ *and all* $t \in (0, T]$, *and*

ii) $\mathrm{Rc}_{g(0)} \geq -\delta_0$ *for some* $\delta_0 > 0$.

Then there exists $\hat{S} = \hat{S}(c_0, \delta_0) > 0$ *such that* $\mathrm{Rc}_{g(t)} \geq -100\delta_0 c_0$ *for all* $0 \leq t \leq \min(\hat{S}, T)$ *on* $B_{g(t)}(x_0, 1)$

Notes on the proof:Our proof involves two steps: Step 1. We consider the evolution of the function $F(x,t) :=$ $\eta(x,t)\mathrm{Rc}(x,t) + [8\delta_0 \mathrm{R}(x,t)t^{\alpha} + t^{\epsilon} + 7\delta_0]g(x,t)$ for some carefully chosen $\alpha, \epsilon \in (0,1)$, where η is a time evolving cut-off function, with $\eta(\cdot) \in [0,1]$ $\eta(\cdot) = 1$ on $B_{g(t)}(x_0, 3/2)$ and $\eta(\cdot) = 0$ on $B_{g(t)}(x_0, 7/4)$. The maximum principle applied to the evolution equation of F shows us that $F > 0$ on $[0,T]$.

Step 2. consider
$F := \eta(x,t)\mathrm{Rc}(x,t) + [8\delta_0 \mathrm{R}(x,t)t + t^{\epsilon} + 7\delta_0]g(x,t)$ (η a cut off on a smaller ball),and use the maximum principle again, with the help of the estimate proved above , to show that this new F also never reaches zero. This implies the result. In both steps an analysis of the term G in the equation $\frac{\partial}{\partial t}F = \Delta_g F + G$ at a first time and point where F potentially has a zero direction is the decisive part of the proof, and is a lengthy algebraic calculation.

The third Lemma we need is a Lemma of R. Hochard, which explains how to conformally change a non-complete Riemannian manifold with unbounded curvature into a complete, non-compact Riemannian manifold of bounded curvature, without changing a region which we wish to preserve.

Lemma 8 (Conformal Lemma,Hochard '16)

Let (N^n, g) *be a smooth (not necessarily complete) Riemannian manifold and let* $U \subset N$ *be an open set. Assume that for some* $\rho \in (0, 1]$, *we have* $\sup_U |\mathrm{Rm}|_g \leq \rho^{-2}$, $B_g(x, \rho) \subset\subset N$ *and* $\mathrm{inj}_g(x) \geq \rho$ *for all* $x \in U$. *Then there exist a constant* $\gamma = \gamma(n) \geq 1$, *an open set* $\tilde{U} \subset U$ *and a smooth metric* $\tilde{g}$ *defined on* $\tilde{U}$ *such that each connected component of* $(\tilde{U}, \tilde{g})$ *is a complete Riemannian manifold satisfying*

i) $\sup_{\tilde{U}} |\mathrm{Rm}|_{\tilde{g}} \leq \gamma\rho^{-2}$

ii) $U_{2\rho} \subset \tilde{U} \subset U$

iii) $\tilde{g} = g \text{ on } U_{2\rho}$, *where* $U_{2\rho} = \{x \in U \mid B_g(x, 2\rho) \subset\subset U\}$.

Idea : Consider the distance function from the boundary of U and mollify this (similar to a method by Cheeger/Gromov 'Chopping Riemannian manifolds'). Then using this smooth function, the strategy of Hochard to prove this lemma is to conformally blow up the metric in a neighbourhood of the boundary of U so that it looks essentially hyperbolic. A similar procedure was used by P. Topping in the case that $n = 2$.

With these ingredients, we are ready to construct a local Ricci-Flow. We recall the three Lemma:

Local Lemma: $v_0 > 0$ with $\mathrm{vol}(B_{g(0)}(x_0, 2)) \geq v_0 > 0$, and $\mathrm{Rc}_{g(t)} \geq -1$ on $B_{g(t)}(x_0, 2)$ for all $t \in [0, T]$ $\implies |\mathrm{Rm}|_{g(t)}(\cdot) \leq \frac{C_0(v_0)}{t}$ and $\mathrm{inj}_{g(t)}(\cdot) \geq \frac{\sqrt{t}}{\sqrt{C_0}}$ on $B_{g(t)}(x_0, 1)$ for $0 < t \leq \min(\hat{T}(v_0), T)$.

DB Lemma: $c_0, \delta_0 > 0$ given with $|\mathrm{Rm}|_{g(t)} \leq \frac{c_0}{t}$ for $t \in (0, T]$, and $\mathrm{Rc}_{g(0)} \geq -\delta_0$ on $B_{g(0)}(x_0, 2)$ $\implies \mathrm{Rc}_{g(t)}(\cdot)) \geq -100\delta_0 c_0$ on $B_{g(t)}(x_0, 1)$ for $t \leq \hat{S} = \hat{S}(c_0, \delta_0) > 0$.

Hochard's Conformal Lemma: Conformally modify regions (Ω, g) with bounded geometry to get $(\tilde{\Omega}, \tilde{g})$ with $\hat{\Omega} \subseteq \tilde{\Omega} \subseteq \Omega$ and $g = \tilde{g}$ on $\hat{\Omega}$ **and** $(\tilde{\Omega}, g)$ is complete with bounded geometry with similar bounds to (Ω, g).

Getting set up

a) W.l.o.g. $\mathrm{Rc}(g_0) \geq -\alpha_0$ on $B_{g_0}(x_0, 3)$, $0 < \alpha_0 << 1$: if not, scale once at $t = 0$

b) W.l.o.g. (M, g_0) has (BC):
If not , apply the Conformal Lemma, keeping g_0 unchanged on $B_{g_0}(x_0, \frac{5}{2})$.

c) W.l.o.g vol$(B_{g(0)}(x, r)) \geq v_0 r^3$ for $x \in B_{g_0}(x_0, \frac{5}{2})$ due to Bishop-Gromov (here v_0 may decrease by a constant factor).

d) Ricci flow of (M, g_0) gives us a (Hamilton/Shi) smooth (BC) solution $(M, g(t))_{t\in[0,t_1]}$ for some small $t_1 > 0$. W.l.o.g $|\mathrm{Rm}(\cdot, t)| \leq \gamma(3)\frac{C_0(v_0)}{t}$ for $t \in [0, t_1]$, where $\gamma = \gamma(3)$ is a constant (to be determined later): the smoothness of the (BC) solution guarantees this.

e) In the following, the distance estimates explained at the beginning will guarantee that $d_t \sim d_0 + \sqrt{at}$ where $a = a(v_0)$, since we will always have $|\mathrm{Rm}(\cdot, t)| \leq \frac{K(v_0)}{t}$. All constants $a(v_0), a_0(v_0), a_1(v_0), \ldots, a_5(v_0)$ are positive and depend only on v_0.

The iteration procedure (analagous to an iteration procedure used by Hochard '16.) We begin with a smooth solution (local) defined on $(B_{g(0)}(x_0, r_1 := 2 - a_4\sqrt{t_1}), g(t))_{t\in[0,t_1]}$ such that $|\mathrm{Rm}(\cdot, t)| \leq \gamma(n)\frac{C_0(v_0)}{t}$ for $t \in [0, t_1]$ and a smooth $(N_1, g(t_1))$ which is (BC) and (CMPL) ($N_1 = M$) with $B_{g(0)}(x_0, r_1) \subset\subset M$ and $|\mathrm{Rm}(g(\cdot, t_1))| \leq \gamma\frac{C_0(v_0)}{t_1}$.
We will construct a local smooth solution defined on $(B_{g(0)}(x_0, r_2 := r_1 - a_4\sqrt{t_2}), g(t))_{t\in[0,t_2]}$ where $t_2 = (1 + \epsilon_0(v_0))t_1$ for an $\epsilon_0(v_0) > 0$, such that $|\mathrm{Rm}(\cdot, t)| \leq \gamma\frac{C_0(v_0)}{t}$ for $t \in [0, t_2]$ and a smooth $(N_2, g(t_2))$ which is (BC) and (CMPL) with $B_{g_0}(x_0, r_2) \subset\subset N_2$ and $|\mathrm{Rm}(g(\cdot, t_2)| \leq \gamma\frac{C_0(v_0)}{t_2}$.

Repeating the iteration procedure i times, we get $t_i = (1+\epsilon_0)^{i-1} t_1$, and we have shrunk the radius to $r_i = 2 - a_4 \sum_{j=1}^{i} \sqrt{t_j} =$
$2 - a_4 \sqrt{t_i} (1 + \frac{1}{\sqrt{1+\epsilon_0}} + \ldots + \frac{1}{\sqrt{1+\epsilon_0}^{i-1}})$
$\geq 2 - a_4 \sqrt{t_i} L_0(v_0)$. Hence if we choose i large (t_1 is small) to be the last i for which, $\sqrt{t_i} \leq \frac{1}{a_4 L_0}$, then we get $t_i = \frac{t_{i+1}}{1+\epsilon_0} > T(v_0) := \frac{1}{(1+\epsilon_0)(a_4 L_0)^2}$ and the flow is defined for $t \in [0, T(v_0)]$ on a ball of radius $\geq 2 - a_4 \sqrt{t_i} L_0(v_0) \geq 2 - 1 = 1$, as required.

The details of the iteration procedure

Step 1. Flow $(N = N_1, g(t_1))$ with the Hamilton/Shi result: we obtain a solution $(N, g(t))_{t\in[t_1,(1+\epsilon_0(v_0))t_1=:t_2]}$ with $|\mathrm{Rm}(\cdot,t)| \leq \frac{4\gamma C_0}{t_2}$ for all $t \in [t_1, t_2]$, and hence $|\mathrm{Rm}(\cdot,t)| \leq \frac{4\gamma C_0}{t}$ for all $t \in [0, t_2]$ on $B_{g(0)}(x_0, r_1)$

Step 2 (Scaled vn. of) the DB Lemma $\implies$ $\mathrm{Rc}(g(\cdot,t)) \geq -100\alpha_0 4C_0\gamma >> -1$ on $B_{g_0}(x_0, r_1 - a_1(v_0)\sqrt{t_2})$ for $t \in [0, t_2]$.

Step 3 (Scaled vn. of) the Local Lemma $\implies$ $|\mathrm{Rm}(\cdot,t)| \leq \frac{C_0}{t}$ and $\mathrm{inj}(g(t))(\cdot) \geq \sqrt{\frac{t}{C_0}}$ on $B_{g_0}(x_0, r_1 - a_2(v_0)\sqrt{t_2})$ for $t \in [0, t_2]$.

Step 4 The Conformal Lemma applied to $(B_{g(0)}(x_0, r_1 - a_2\sqrt{t_2}), g(t_2))$ leads to a smooth (CMPL), (BC) $(N_2, g(t_2))$ where $g(t_2)$ is the original one (unchanged) on $B_{g(0)}(x_0, 2 - a_3\sqrt{t_2})$ with $|\mathrm{Rm}(\cdot, t_2)| \leq \frac{\gamma C_0}{t_2}$ and a local solution $(B_{g(0)}(x_0, 2 - a_4\sqrt{t_2}), g(t))_{t\in[0,t_2]}$ with $B_{g(0)}(x_0, 2 - a_4\sqrt{t_2}) \subset\subset N_2$.

In fact, we see in the iteration procedure, that we can only do this procedure as long as $t_i \leq \hat{T}(v_0), \hat{S}(v_0)$ appearing in the the statement of the DB Lemma resp. Local Lemma. This is a further, non-harmful, constraint on the time.

Also, our proof gives us more than the ACCT conjecture. Using the Hölder distance estimates explained above, we get...

Theorem 9 (ACCT Si.+Topping '17)

Given (M_i^3, g_i, y_i) *CMPL such that* $(i)\mathrm{Rc}_{g_i} \geq -1$ *on* M_i *and* $(ii)\,\mathrm{vol}(B_{g_i}(y_i, 1)) \geq v_0 > 0$. *Then there exist a three-dimensional topological manifold* M *and a metric* $d : M \times M \to [0, \infty)$ *generating the same topology as* M *and making* (M, d) *a complete metric space, such that after passing to a subsequence, we have* $(M_i, d_{g_i}, y_i) \to (M, d, y_0)$ *in the pointed Gromov-Hausdorff sense, for some* $y_0 \in M$*, and...*

For any $B_d(y_0, R) \subset M$ we can find $\alpha < 1,\ C > 0$ depending on R and v_0 and maps $\varphi_i : B_d(y_0, R) \to M_i$ which are homeomorphisms onto their images, such that each φ_i is an $\epsilon(i)$ Gromov-Hausdorff approximation, with $\epsilon(i) \to 0$ as $i \to \infty$ which is Hölder in the following sense:

$$\frac{1}{C} d_i^{\frac{1}{\alpha}}(\varphi_i(x), \varphi_i(y)) \le d(x, y) \le C d_i^{\alpha}(\varphi_i(x), \varphi_i(y)) \quad (1)$$

Open problem Can one construct a local Ricci flow of this type in case (5) (Kähler and bi-sectional curvature not less than minus one), with good estimates, as in case (1)-(4).

Open Problem: Do local Pseudolocality results of this type exist for the other curvature conditions mentioned?

Open Problem: Can one remove the (BC) condition in the 3d Pseudolocality result?

Note: For $n = 2$ we can remove (BC) and (CMPL) if we include $B_{g(t)}(x_0, 4) \subset\subset M$ for all $t \in [0, T]$ (Si. '13).